APPLICATION OF INTEGRALS

APPLICATION OF INTEGRATION TO AREAS

Definite integral is used to evaluate areas bounded by curves. To do problems under this heading, one must be able to draw a rough figure of the curve when the equation is given. Some rules about drawing curves are given below. Familiar curves like lines, circles and conics are not discussed here.

Guidelines

(i) Check whether the curve is symmetrical about the x-axis or not. The curve is symmetrical about the x-axis, if its equation is unchanged when y is replaced by $-y$.

(ii) The curve is symmetrical about the y-axis if its equation is unchanged when x is replaced by $-x$.

(iii) Put $y = 0$ in the equation of the curve. This will give the points where it cuts the x-axis

(iv) Put $x = 0$ in the equation of the curve. This will give the points where it cuts the y-axis.

(v) The curve is symmetrical about the line $y = x$ if its equation does not change when x and y are interchanged.

(vi) Find the turning points of the graph by equating $\frac{dy}{dx} = 0$

(vii) Find the intervals of curve in which it increases and decreases if required.

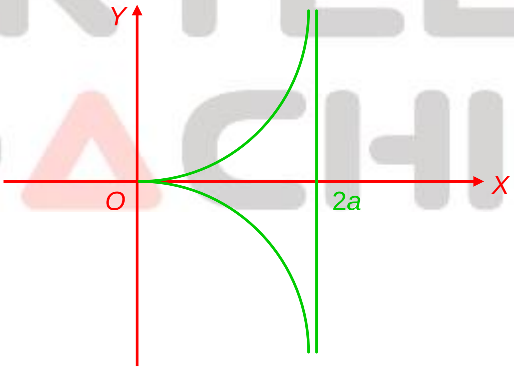
(viii) Use periodicity wherever possible.

(ix) Check behaviour at $x \to \pm\infty$ and $y \to \pm\infty$.

Illustration 1

Question: **Trace the curve $y^2(2a - x) = x^3$, $a > 0$.**

Solution: Note that the curve passes through the origin and is symmetrical about the x-axis. $y^2 = \frac{x^3}{2a - x}$

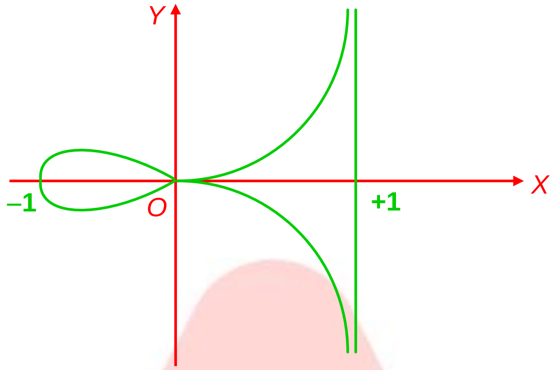


L.H.S. is positive. If x is negative or if x is greater than $2a$, R.H.S. becomes negative. Hence the curve lies only in the interval 0 to $2a$. When $x \to 2a$, $y \to \infty$. Therefore the line $x = 2a$ is an asymptote for the curve. A rough Figure is shown.

Illustration 2

Question: **Trace the curve $y^2 = \frac{x^2(1+x)}{1-x}$**

Solution: The curve passes through the origin and is symmetrical about the x-axis. It intersects the x-axis at $x = -1$ and $x = 0$. If $x < -1$ or $x > 1$ the curve is non-existent. As $x \to 1$, $y \to \pm\infty$ a rough diagram is shown below.

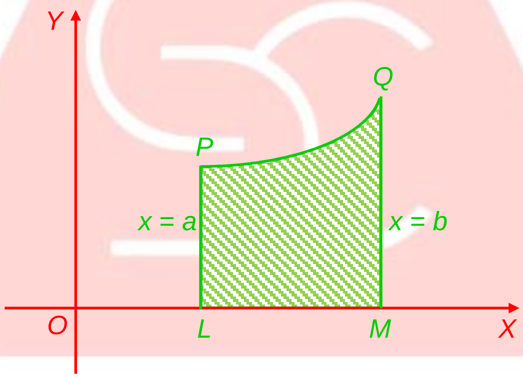


The curve has a loop between – 1 and 0.

ESTIMATION OF AREAS

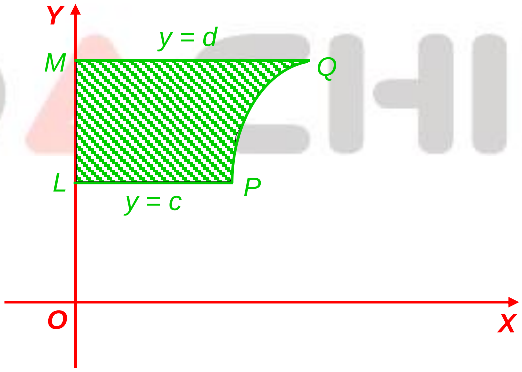
Four cases are discussed below:

Case I : PQ is an arc of a curve whose equation is $y = f(x)$. We have an area bounded by PQ on one side; by the x-axis on another and the two parallel lines $x = a$ and $x = b$ (shown by PL and QM), $a < b$.



The area $PLMQ = \int_{x=a}^{x=b} y\,dx = \int_a^b f(x)\,dx$

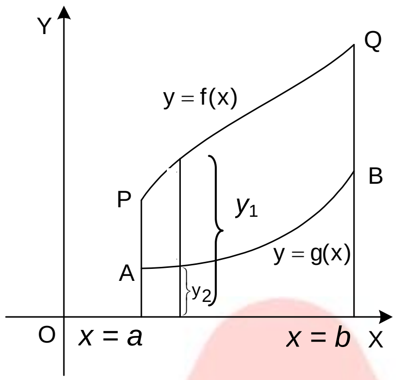
Case II: PQ is an arc of a curve whose equation is $y = f(x)$ or $x = f(y)$.
In this case y-axis is one boundary and the other two are the lines $y = c$ and $y = d$.



The area $LPQM = \int_{y=c}^{y=d} x\,dy = \int_c^d f(y)\,dy$

In this case the integration is with respect to y.

Case III: The figure encloses an area between two curves one of which is represented by PQ with equation $y = f(x)$ and the other by AB with the equation $y = g(x)$.

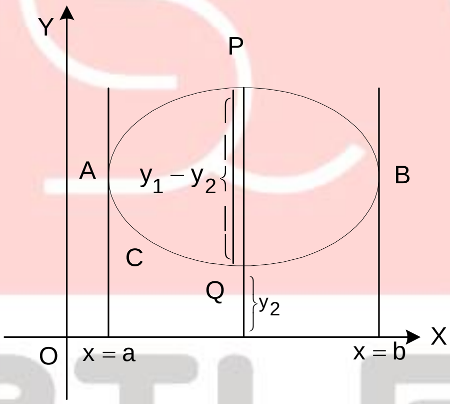


Area $PABQ = \int_a^b (y_1 - y_2)\,dx$ where $y_1 = f(x)$ and $y_2 = g(x)$

$$= \int_a^b \{f(x) - g(x)\}\,dx$$

Case IV: The figure represents the region bounded by a closed curve ACQBP.

The area of the region bounded by a closed curve ACQBP is $\int_a^b (y_1 - y_2)\,dx$, $y_1 > y_2$

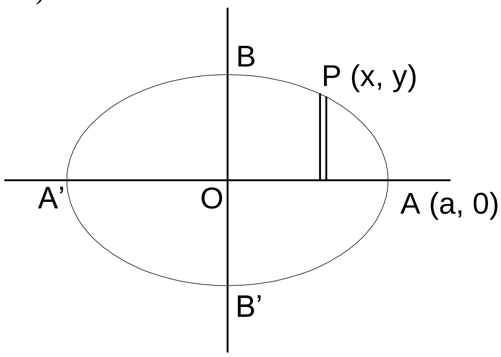


The values of y_1 and y_2 are obtained by solving the equation of the curve as a quadratic in y whose larger root y_1 and smaller root y_2 are functions of x.
a and b are the coordinates of the points of contact of tangents drawn parallel to the y-axis.

Illustration 3

Question: **Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$**

Solution: The ellipse is symmetrical about both axes and hence the area enclosed
= 4 (area of the quadrant)



$$= 4\int_0^a y\,dx$$

$$= 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{(a^2-x^2)}\,dx = \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

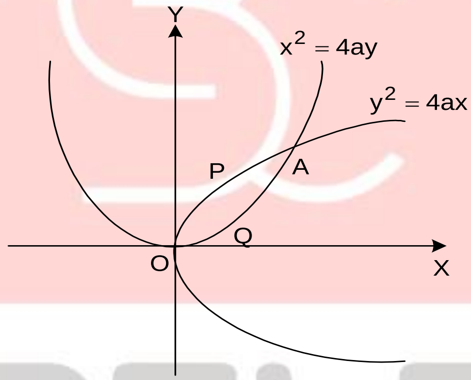
$$= \frac{4b}{a}\left[\frac{a^2\pi}{4}\right] = \pi ab \text{ sq. units}$$

Illustration 4

Question: **Find the area included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.**

Solution: The two parabolas intersect at O(0,0) and A (4a, 4a). The area included between the two curves = area OQAP.

$$= \int_{x=0}^{x=4a} (y_1 - y_2)\,dx$$



$$= \int_0^{4a}\left(2\sqrt{a}\,\sqrt{x}-\frac{x^2}{4a}\right)dx$$

$$= \left[2\sqrt{a}\,\frac{2}{3}x^{3/2}-\frac{x^3}{12a}\right]_0^{4a}$$

$$= 4\frac{\sqrt{a}}{3}8.a^{3/2}-\frac{64a^3}{12a}$$

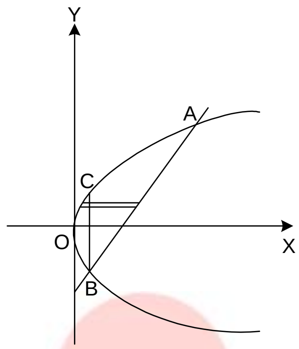
$$= \frac{16a^2}{3} \text{ sq.units.}$$

Note: Sometimes it is better to use the formula $\int_c^d x\,dy$ instead of $\int_a^b y\,dx$ in the computation of area to simplify calculations, as the following illustration shows.

Illustration 5

Question: **Find the area of the segment cut off from the parabola $y^2 = 2x$ by the line $y = 4x - 1$.**

Solution: The line $y = 4x - 1$ intersects the parabola $y^2 = 2x$ at A and B



$$2x = (4x-1)^2 \Rightarrow 16x^2 - 10x + 1 = 0$$

$$\Rightarrow \quad (8x-1)(2x-1) = 0$$

$$\therefore \quad A = \left(\frac{1}{2}, 1\right) \text{ and } B = \left(\frac{1}{8}, -\frac{1}{2}\right)$$

If the formula $\int y\,dx$ is to be used then the area will have to be split up as *OBC* and *CBA*. Instead the problem can be done directly by using the formula $\int (x_2 - x_1)\,dy$.

$$\text{Area required } = \int_{y=-1/2}^{1} (x_2 - x_1)\,dy$$

$$= \int_{-1/2}^{1} \left(\frac{y+1}{4} - \frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^2}{8} + \frac{y}{4} - \frac{y^3}{6}\right]_{-1/2}^{1}$$

$$= \left(\frac{1}{8} + \frac{1}{4} - \frac{1}{6}\right) - \left(\frac{1}{32} - \frac{1}{8} + \frac{1}{48}\right)$$

$$= \frac{(3+6-4)}{24} - \frac{(3-12+2)}{96} = \frac{5}{24} + \frac{7}{96} = \frac{27}{96} = \frac{9}{32} \text{ sq. units}$$

MIND MAP

AREAS BOUNDED BY CURVES

Algebraic area bounded by

$y = f(x)$, $x = a$, $x = b$ and $y = 0$

is given by $\int_a^b f(x)\,dx$.

When the curve is above x-axis the value of definite integral is positive and when it is below, the value is negative.

Algebraic area bounded by

$y = g(y)$, $y = c$, $y = d$ and $x = 0$

is given by $\int_c^d g(y)\,dy$.

When the curve is to the right side of y-axis the value of definite integral is positive and when it is on the left side the value is negative.

Area bounded by

$y = f(x)$, $y = g(x)$, $x = a$ and $x = b$

is given by $\int_a^b [f(x) - g(x)]\,dx$, irrespective of the position of x-axis. ($f(x) > g(x)$)

Area bounded by

$x = f(y)$, $x = g(y)$, $y = c$ and $y = d$

is given by $\int_c^d [f(y) - g(y)]\,dy$, irrespective of the position of y-axis. ($f(y) > g(y)$)

Important points for curve sketching

- Permissible values of x and y.
- Symmetricity w.r.t. coordinate axes and the line $y = x$.
- Points of intersection with coordinate axes.
- Turning points / critical points.
- Intervals of monotonicity.
- Periodicity
- Behaviour at $x \to \pm\infty$ and $y \to \pm\infty$.
- Knowledge of some basic curves.