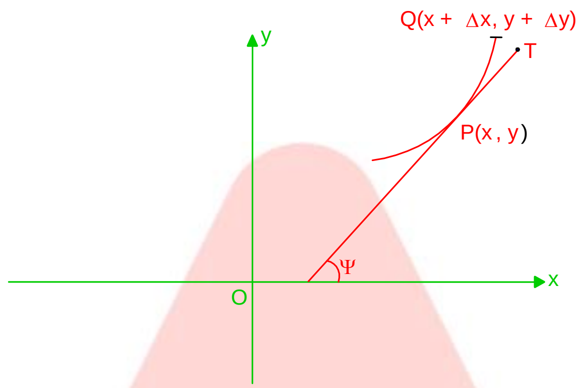
APPLICATION OF DERIVATIVES

1 GEOMETRICAL INTERPRETATION OF DERIVATIVE

Consider a curve $y = f(x)$ and two points $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ on it. Then $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$



$= \lim_{\Delta x \to 0} \left(\frac{y + \Delta y - y}{x + \Delta x - x} \right)$

$= \lim_{\Delta x \to 0}$ (slope of the chord PQ)

= Slope of the tangent PT at $P(x, y)$

= $\tan \psi$, where ψ is the angle which the tangent at P makes with the positive direction of the x-axis.

2 TANGENT AND NORMAL

Tangent at a point is a line which touches the curve at that point and normal at a point is a line which is perpendicular to the tangent at that point.

Given the equation of a curve $y = f(x)$ and a point A (x_1, y_1) on it, the equation of the tangent at A is

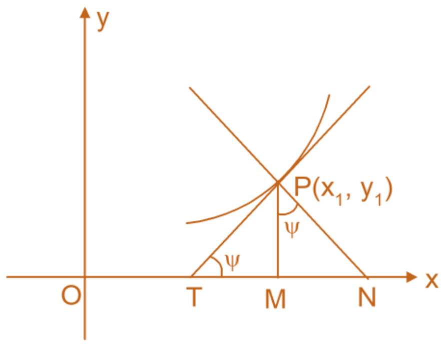
$$y - y_1 = \left(\frac{dy}{dx} \right)_{at A} (x - x_1)$$

and the equation of the normal at A is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx} \right)_{at A}} (x - x_1)$$

- **Length of tangent, normal, subtangent and subnormal**

Let the tangent and normal at $P(x, y)$ meet the x-axis at T and N respectively. PT is called the length of the tangent and is equal to PM cosec ψ $= \frac{y_1}{\sin \psi}$ (from Δ TMP)



Hence length of tangent (PT)

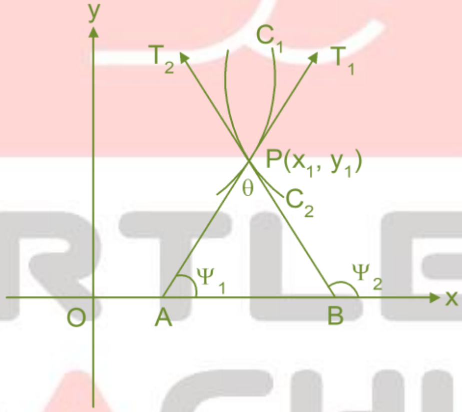
$$= \left|\frac{y_1 \sec\psi}{\tan\psi}\right| = \left|\frac{y_1\sqrt{1+m^2}}{m}\right| \text{ where } m = \left(\frac{dy}{dx}\right)$$

Similarly, the following results can easily be concluded.

- PN is called the length of the normal = $PM \sec \psi$ = $\left|y_1\sqrt{1+m^2}\right|$ (from ΔMNP)
- TM is called the subtangent = $PM. \cot \psi = \left|\frac{y_1}{\tan\psi}\right| = \left|\frac{y_1}{m}\right|$ (from ΔTMP)
- MN is called the subnormal = $PM \tan \psi = | y_1 m |$ (from ΔMNP)

3 ANGLES BETWEEN TWO CURVES

Given two curves $C_1 : y = f(x)$ and $C_2 : y = g(x)$ intersecting at some point $P(x_1, y_1)$.



Let PT_1 be the tangent at P to curve C_1 and let PT_1 make an angle ψ_1 with OX. Let PT_2 be the tangent at P to curve C_2 and let PT_2 make an angle ψ_2 with OX. The angle between two curves is defined to be the angle between the two tangents at the point of intersection.

∴ θ (the angle between the curves)

$\angle APB = T_1PT_2 = \psi_2 - \psi_1$ from ΔABP

$$\tan\theta = \tan(\psi_2 - \psi_1) = \frac{\tan\psi_2 - \tan\psi_1}{1+\tan\psi_2\tan\psi_1} = \frac{m_2 - m_1}{1+m_1m_2}$$

If θ is the acute angle between the two curves

$$\tan\theta = \left|\frac{m_2 - m_1}{1+m_1\,m_2}\right| \text{ where } m_1 = f'(x) \text{ at } P \text{ and } m_2 = g'(x) \text{ at } P.$$

Remark:

(i) curves intersect orthogonally if $m_1 m_2 = -1$;

(ii) curves touch each other if $m_1 = m_2$.

4 dy/dx AS RATE MEASURER

This article shows, how $\frac{dy}{dx}$ is useful in determination of rates of change related to physical situations.

5 ERRORS AND APPROXIMATIONS

Let $y = f(x)$. Then, we know that $\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x)$.

$\therefore$ $\frac{f(x+\delta x)-f(x)}{\delta x} = f'(x)+\varepsilon$, where $\varepsilon \to 0$, when $\delta x \to 0$.

So, $f(x+\delta x)-f(x) = f'(x).\delta x + \varepsilon\,.\delta x$

or $f(x+\delta x)-f(x) = f'(x).\,\delta x$ (approximately)

or $\delta y = f'(x).\delta x$ $[\because\ f(x+\delta x)-f(x)=\delta y]$

Thus, if δx is an error in x, then the corresponding error δy in y can be calculated.

- δx and δy are known as differentials.
- If $y = f(x)$ and δy is an increment in y, corresponding to an increment δx in x, then we have $\delta y = \frac{dy}{dx}.\delta x$.
- The error δx in x is called an absolute error.
- $\frac{\delta x}{x}$ is called the relative error and $\left(\frac{\delta x}{x}\times 100\right)$ is called the percentage error.

6 MONOTONICITY OF FUNCTIONS

In this section we shall study the behaviour of functions. Basically we have four kinds of behaviours shown in functions in intervals of their domains.

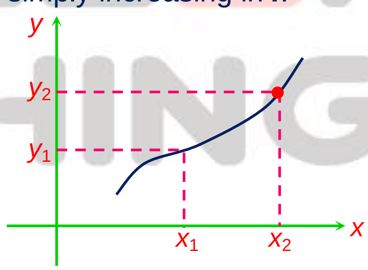
6.1 INCREASING BEHAVIOUR

If in an interval ***I***, for any two points (x_1, y_1) and (x_2, y_2) we have

$x_2 > x_1 \Leftrightarrow y_2 > y_1$

the function is said to be monotonically increasing or simply increasing in ***I***.

If the function is differentiable in the interval of interest (which is normally true for most of the functions), it can be inferred that $\frac{dy}{dx} > 0$ for all points in that interval.



e.g., $y = x^3$

$\Rightarrow$ $\frac{dy}{dx} = 3x^2$

Now $\frac{dy}{dx} > 0$ for all real x except $x = 0$.

$\therefore$ Here $\frac{dy}{dx} \ge 0$ for the entire domain but still the function is increasing. For any two points such that $x_2 > x_1$, we have $y_2 > y_1$ certainly.

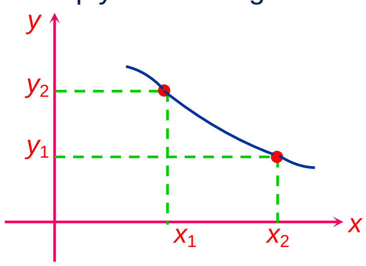
6.2 DECREASING BEHAVIOUR

If in an interval ***I***, for any two points (x_1, y_1) and (x_2, y_2) we have

$$x_2 > x_1 \quad \Leftrightarrow \quad y_2 < y_1$$

the functions is said to be monotonically decreasing or simply decreasing in ***I***.

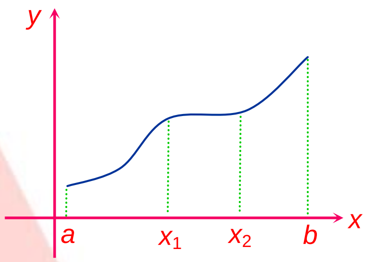
Again for differentiable function, here $\frac{dy}{dx} < 0$ for all points in the interval.



6.3 NON-DECREASING BEHAVIOUR

Here $x_2 > x_1 \Leftrightarrow y_2 \geq y_1$, for all points in that interval.

$\Rightarrow \frac{dy}{dx} \geq 0$, where $\frac{dy}{dx} = 0$ for a continuous set of points in the interval.



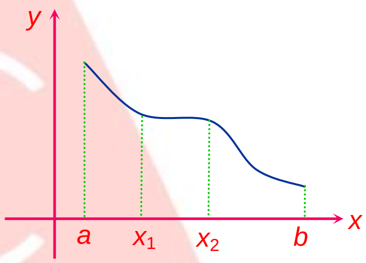
6.4 NON-INCREASING BEHAVIOUR

In this case,

$$x_2 > x_1 \quad \Leftrightarrow \quad y_2 \leq y_1$$

for all points in that interval.

$\Rightarrow \frac{dy}{dx} \leq 0$, where $\frac{dy}{dx} = 0$ for a continuous set of points in the interval.



Caution: These conditions normally sufficient to find intervals of increase but sometimes the derivative may be zero also at specific points in the interval and still the function may be increasing.

7 MAXIMA AND MINIMA OF FUNCTIONS OF A SINGLE VARIABLE

Definition: Let $f(x)$ be defined on an interval A. Let a and $b \in A$. Then

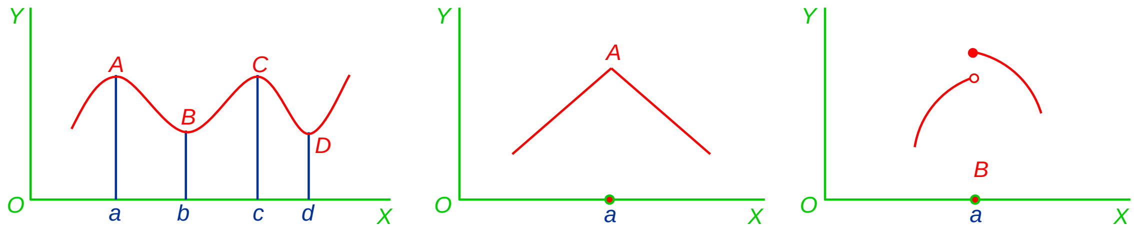
(i) $f(x)$ is said to have a maximum value at $x = a$ if $f(a) > f(a + h)$ and $f(a) > f(a - h)$ for all sufficiently small positive values of h. The point a is called the point at which the function is a maximum and $f(a)$ is the corresponding maximum value of the function.

(ii) $f(x)$ is said to have a minimum value at $x = b$ if $f(b) < f(b + h)$ and $f(b) < f(b - h)$ for all sufficiently small positive values of h. The point b is called the point at which the function is a minimum.

Maximum and minimum values of $f(x)$ as defined above are not necessarily the greatest and least values of $f(x)$. They are maximum and minimum in the immediate neighbourhood of $x = a$ and $x = b$. Hence these are also referred as local maximum or local minimum.

The points of maximum or minimum of a function are also called the points of extremum.

A necessary condition for the existence of an extremum (maximum or minimum) for a function $f(x)$ is either $f'(x) = 0$ or $f'(x)$ does not exist.



In the first figure, at the maximum and minimum points on the graph, the tangent is parallel to the x-axis and hence f ′(x) = 0. In the second figure, the function is increasing as x approaches a from the left and is decreasing as x increases beyond a. The graph is not smooth and hence has no tangent at x = a. In the third figure, x = a is a point of local maxima ($f(x)$ is discontinuous at x = a). It is very important to note that for maxima and minima we use basic definition at all the critical points (The points where $f(x)$ is discontinuous).

7.1 SECOND DERIVATIVE TEST (For Continuous Functions)

- If $f(x)$ has a maximum value at $x = a$ and $f'(a)$ exists, then $f'(a)$ must be zero. Similarly, if $f(x)$ has a minimum value at $x = b$ and $f'(b)$ exists, then $f'(b)$ must be zero.
- If c be a point in the domain of $f(x)$ such that
  $f'(c) = 0$ and $f''(c) \neq 0$, then
  - $x = c$ is a point of local maxima if $f''(c) < 0$
  - $x = c$ is a point of local minima if $f''(c) > 0$

7.2 FIRST DERIVATIVE TEST (For Continuous Functions)

- If $f'(a) = 0$ or does not exist and $f'(x)$ changes sign from plus to minus as x passes through the point 'a' from left to right, then $f(x)$ is maximum at $x = a$.
- If $f'(b) = 0$ or does not exist and $f'(x)$ changes sign from minus to plus as x passes through the point 'b' from left to right, then $f(x)$ is minimum at $x = b$.

If the derivative does not change sign in moving from left to right through the point a, then $f(x)$ has neither maximum nor minimum at $x = a$.

7.3 n^{th} DERIVATIVE TEST

(It can be applied to $x = c$ only if $f'(c) = 0$ and $f''(c) = 0$).
By differentiation, find nth derivative of $f(x)$ denoted by $f^n(x)$, $n \in \mathbf{N}$
Step-by-step, find the earliest non-zero $f^n(c)$, $n = 3, 4, 5, 6, 7, \ldots\ldots\ldots$
In this process,
(i) if n is odd $\Rightarrow$ $x = c$ is neither local maximum nor local minimum point.
(ii) if n is even, and if

$$f^n(c) = \begin{cases} +ve \ , & \Rightarrow \ x = c \text{ is local minimum point} \\ -ve \ , & \Rightarrow \ x = c \text{ is local maximum point.} \end{cases}$$

7.4 ABSOLUTE MAXIMUM / MINIMUM POINTS

- To find absolute maximum / minimum values of $f(x)$ in open interval (a, b), we proceed as follows:
  - Find all extremum points of $f(x)$ by using critical points. Let these extremum points be c_1, c_2, c_3 ………..
  - Compare the lengths of ordinates $f(c_1)$, $f(c_2)$, $f(c_3)$, ………….
  - The greatest value of these ordinates is called absolute maximum value of $f(x)$.
  - The least value of these ordinates is called absolute minimum value of $f(x)$.

  Absolute maximum value/absolute minimum value can occur at more than one extremum point. Absolute maximum/minimum value is also called Global maximum/minimum value of $f(x)$.
- To find absolute maximum/minimum value in closed interval $[a, b]$, include values of ordinates at the end points viz. $f(a)$ and $f(b)$, in the above procedure of comparison of lengths of the ordinates at the extremum points.

8.1 ROLLE'S THEOREM

If $f(x)$ is continuous in the interval $[a, b]$ and differentiable in (a, b) and further $f(a) = f(b)$, then there is at least one point $x = c$ on the interval (a, b) where $f'(c) = 0$.

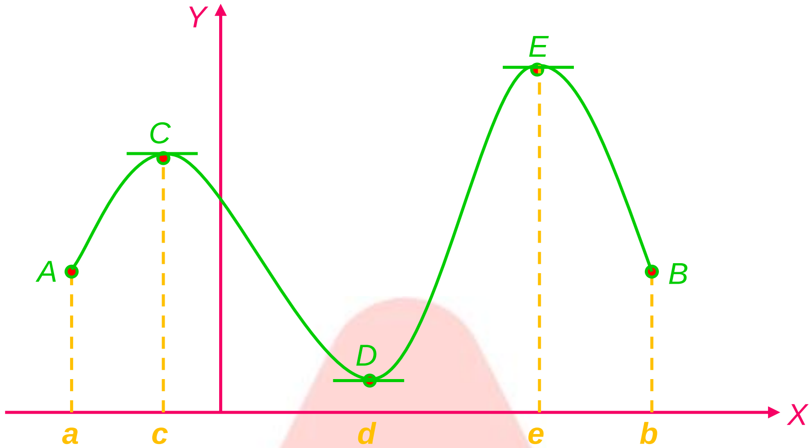


Figure shows the graphical interpretation of Rolle's theorem. The slope of tangent is zero at the points C, D and E.

9 PRACTICAL PROBLEMS

In this section, we will use differential calculus in optimization problems.

MIND MAP

APPLICATIONS OF DIFFERENTIAL CALCULUS

- Slope of tangent at $x = a$ on $y = f(x)$ is $\left.\frac{dy}{dx}\right|_{x=a}$ or $f'(a)$.
- Equation of tangent at $P(x, y)$ on the curve : $Y - y = \frac{dy}{dx}(X - x)$
- Equation of normal at $P(x, y)$ on the curve : $Y - y = -\frac{dx}{dy}(X - x)$

- Acute angle between curves $y = f(x)$ and $y = g(x)$ at (x_1, y_1) :

$$\tan\theta = \left|\frac{f'(x_1) - g'(x_1)}{1 + f'(x_1)g'(x_1)}\right|$$

Curves touch if $f'(x_1) = g'(x_1)$

they are orthogonal if $f'(x_1) \cdot g'(x_1) = -1$

- Relative error : $\frac{\delta x}{x}$
- Percentage error : $\left(\frac{\delta x}{x} \times 100\right)$
- If $y = f(x)$ and δy is an increment in y, correspoinding to an increment δx in x, then we have $\delta y = dy/dx . \delta x$

Monotonicity of functions

- Increasing : $x_2 > x_1 \Leftrightarrow y_2 > y_1$
  $f'(x) > 0$
- Decreasing : $x_2 > x_1 \Leftrightarrow y_2 < y_1$
  $f'(x) < 0$
- Non-decreasing : $x_2 > x_1 \Leftrightarrow y_2 \geq y_1$
  $f'(x) \geq 0$, where $f'(x) = 0$ for continuous set of points in the interval.
- Non-increasing : $x_2 > x_1 \Leftrightarrow y_2 \leq y_1$
  $f'(x) \leq 0$, where $f'(x) = 0$ for continuous set of points in the interval.

Points of extrema (critical points):

- Local maxima at $x = a$ if
  $f(a) > f(x)\ \forall\ x \in (a - \epsilon, a) \cup (a, a + \epsilon)$
- Local minima at x = a if
  $f(a) < f(x)\ \forall\ x \in (a - \epsilon, a) \cup (a, a + \epsilon)$
- $f'(x) = 0$ or does not exist at critical points.

First derivative test:

- At point of local maxima $f'(x)$ changes sign from positive to negative.
- At point of local minima $f'(x)$ changes sign from negative to positive.

Second derivative test:

- $f''(a)$ is negative for local maxima.
- $f''(a)$ is positive for local minima.

n^{th} derivative test:

- If $f''(a) = f'''(a) = \ldots\ldots = f^{n-1}(a) = 0$
  and $f^{n}(a) \neq 0$.
- If n is odd, neither maxima nor minima at $x = a$.
- If n is even, $f^{n}(a) < 0 \Rightarrow$ local maxima and $f^{n}(a) > 0 \Rightarrow$ local minima.

Rolle's Theorem:

- If f is continuous in $[a, b]$ and differentiable in (a, b) and $f(a) = f(b)$ then $\exists$ at least one $c \in (a, b)$ such that $f'(c) = 0$.

Lagrange's Mean Value Theorem:

- If f is continuous in $[a, b]$ and differentiable in (a, b) then $\exists$ at least one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$