Binomial Theorem

1 BINOMIAL EXPRESSION

Any algebraic expression consisting of only two terms is known as binomial expression. The terms may consist of variables x, y etc. or constants or their mixed combinations. For example: $2x + 3y$, $4xy + 5$ etc.

2 BINOMIAL THEOREM FOR POSITIVE INDEX

Binomial theorem gives a formula for the expansion of a binomial expression raised to any positive integral power.

In general for a positive integer n

$(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \ldots.. + {}^nC_n x^0 y^n$, where ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$

for $r = 0, 1, 2, \ldots\ldots, n$ is called binomial coefficient.

2.1 PROOF OF BINOMIAL THEOREM

The Binomial theorem can be proved by mathematical induction

Let $P(n)$ stands for the mathematical statement

$$(x + a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + {}^nC_r x^{n-r} a^r + \ldots + a^n \quad \ldots(i)$$

Note that there are $(n+1)$ terms in R.H.S. and all the terms are of the same degree in x and a together.

When $n = 1$, L.H.S. $= x + a$ and R.H.S. $= x + a$ (there are only 2 terms)

$\therefore$ P (1) is verified to be true

Assume $P(m)$ to be true

i.e., $(x + a)^m = x^m + {}^mC_1 x^{m-1} a + {}^mC_2 x^{m-2} a^2 + \ldots + {}^mC_r x^{m-1} a^r + \ldots + a^m \quad \ldots(ii)$

Multiplying equation (ii) by $(x + a)$, we have

$$(x + a)^m (x + a) = (x + a)\left\{x^m + {}^mC_1 x^{m-1} a + {}^mC_2 x^{m-2} a^2 + \ldots + {}^mC_r x^{m-r} a^r + \ldots + a^m\right\}$$

i.e., $(x + a)^{m+1} = x^{m+1} + \left({}^mC_1 + 1\right)x^m a + \left({}^mC_2 + {}^mC_1\right)x^{m-1} a^2 + \ldots$
$+ \left({}^mC_r + {}^mC_{r-1}\right)x^{m-r+1} a^r + \ldots + a^{m+1}$

$$= x^{m+1} + {}^{(m+1)}C_1 x^m a + {}^{(m+1)}C_2 x^{m-1} a^2 + \ldots$$
$$+ {}^{m+1}C_r x^{m+1-r} a^r + \ldots a^{m+1} \quad \ldots(iii)$$

(using the formula ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$)

Equation (iii) implies that $P(m+1)$ is true and hence by induction $P(n)$ is true.

Alternative method

By choosing x from all the brackets we get the term x^n. Choosing x from $(n-1)$ factors and 'a' from the remaining factor we get x^{n-1} a. But the number of ways of doing this is equal to the number of ways of choosing one factor from n factors (i.e.,) ${}^nC_1$. Choosing x from $(n-2)$ factor and a from the remaining two factors, we get $x^{n-2} a^2$. But the number of ways of doing this is equal to the number of ways of choosing two factors from n factors. i.e., ${}^nC_2$. Finally choosing 'a' from all the factors we get the term a^n.

$$\therefore (x+a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \dots + {}^nC_r x^{n-r} a^r + \dots + a^n$$

Illustration 1

Question: **Expand $\left(x - \frac{1}{x}\right)^6$**

Solution: $\left(x - \frac{1}{x}\right)^6 = {}^6C_0 x^6 + {}^6C_1 x^5 \left(\frac{-1}{x}\right) + {}^6C_2 x^4 \left(\frac{-1}{x}\right)^2 + {}^6C_3 x^3 \left(\frac{-1}{x}\right)^3 + {}^6C_4 x^2 \left(\frac{-1}{x}\right)^4 + {}^6C_5 x \left(\frac{-1}{x}\right)^5 + {}^6C_6 x^0 \left(\frac{-1}{x}\right)^6$

$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$

3 GENERAL TERM IN THE BINOMIAL EXPANSION

The general term in the expansion of $(x + y)^n$ is $(r + 1)^{th}$ term, given by $t_{r+1} = {}^nC_r x^{n-r} y^r$ where r = 0, 1, 2 ……… n.

- Every term in the expansion is of nth degree in variables x and y.
- The total number of terms in the expansion is $n + 1$.
- Binomial expansion can also be expressed as $(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$.

Illustration 2

Question: **Find the 11th term in the expansion of $\left(3x - \frac{1}{x\sqrt{3}}\right)^{20}$.**

Solution: The general term $= t_{r+1} = (-1)^r \; {}^{20}C_r \; (3x)^{20-r} \left(\frac{1}{x\sqrt{3}}\right)^r$

For the 11th term, we must take $r = 10$

$\therefore t_{11} = t_{10+1} = (-1)^{10} \; {}^{20}C_{10} \; (3x)^{20-10} \left(\frac{1}{x\sqrt{3}}\right)^{10}$

$= {}^{20}C_{10} \; 3^{10} \; x^{10} \; \frac{1}{x^{10} \; (\sqrt{3})^{10}} = {}^{20}C_{10} \left(\sqrt{3}\right)^{10} = {}^{20}C_{10} \; 3^5$

Illustration 3

Question: **The 2nd, 3rd and 4th terms of $(x + y)^n$ are 240, 720 and 1080 respectively. Find x, y and n.**

Solution: $t_2 = {}^nC_1 x^{n-1} y = 240$

$t_3 = {}^nC_2 x^{n-2} y^2 = 720$

$t_4 = {}^nC_3 x^{n-3} y^3 = 1080$

$\frac{{}^nC_2 x^{n-2} y^2}{{}^nC_1 x^{n-1} y} = \frac{720}{240}$; i.e., $\frac{n-1}{2} \frac{y}{x} = 3$ … (i)

$\frac{{}^nC_3 x^{n-3} y^3}{{}^nC_2 x^{n-2} y^2} = \frac{1080}{720}$; i.e., $\frac{n-2}{3} \frac{y}{x} = \frac{3}{2}$ … (ii)

Dividing equation (i) by equation (ii) we have

$\frac{3(n-1)}{2(n-2)} = 2 \quad \Rightarrow n = 5$

Using $n = 5$ in (i) we have $\frac{y}{x} = \frac{3}{2}$

Substituting in t_2 we have ${}^5C_1\, x^4\, \frac{3}{2}\, x = 240$

$\therefore$ $x^5 = 32 \Rightarrow x = 2$ and $y = 3$

$\therefore$ $x = 2$, $y = 3$ and $n = 5$.

Illustration 4

Question: **Find the term independent of x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$**

Solution: The general term $= {}^9C_r \left(\frac{3x^2}{2}\right)^{9-r} \left(\frac{-1}{3x}\right)^r$

$$= (-1)^r\ {}^9C_r\, \frac{3^{9-2r}}{2^{9-r}} \cdot x^{18-3r}$$

The term independent of x, (or the constant term) corresponds to x^{18-3r} being

x^0 or $18 - 3r = 0 \Rightarrow r = 6$

$\therefore$ the term independent of x is the 7th term and its value is

$$(-1)^6\ {}^9C_6\, \frac{3^{9-12}}{2^{9-6}} = {}^9C_3\, \frac{3^{-3}}{2^3} = \frac{9.8.7}{3.2.1} \cdot \frac{1}{(6)^3} = \frac{7}{18}$$

4 MIDDLE TERMS OF THE EXPANSION

In the binomial expansion of $(x + y)^n$

4.1 WHEN *n* IS ODD

There are $(n + 1)$ i.e. even terms in the expansion and hence two middle terms are given by

$$t_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}}\, x^{\frac{n+1}{2}}\, y^{\frac{n-1}{2}} \quad \text{for} \quad r = \frac{n-1}{2}$$

and $$t_{\frac{n+3}{2}} = {}^nC_{\frac{n+1}{2}}\, x^{\frac{n-1}{2}}\, y^{\frac{n+1}{2}} \quad \text{for} \quad r = \frac{n+1}{2}$$

4.2 WHEN *n* IS EVEN

There are odd terms in the expansion and hence only one middle term is given by

$$t_{\frac{n}{2}+1} = {}^nC_{n/2}\, x^{n/2}\, y^{n/2} \quad \text{for} \quad r = \frac{n}{2}$$

Illustration 5

Question: **Find the middle term in the expression of $(1 - 2x + x^2)^n$.**

Solution: $(1 - 2x + x^2)^n = [(1 - x)^2]^n = (1 - x)^{2n}$

Here $2n$ is even integer, therefore, $\left(\frac{2n}{2} + 1\right)^{th}$ i.e. $(n + 1)^{th}$ term will be the middle term.

Now $(n + 1)^{th}$ term in $(1 - x)^{2n} = {}^{2n}C_n\, (1)^{2n-n}\, (-x)^n$

$$= {}^{2n}C_n(-x)^n = \frac{(2n)!}{n!\,n!}(-x)^n.$$

Illustration 6

Question: **Prove that middle term in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1.3.5.....(2n-1)}{n!}.2^n$.**

Solution: Since $2n$ is even, therefore $\left(\frac{2n}{2}+1\right)^{th}$ i.e. $(n + 1)^{th}$ term will be the middle term.

Now $(n + 1)^{th}$ term i.e. middle term in $\left(x+\frac{1}{x}\right)^{2n}$ is given by

$$t_{n+1} = {}^{2n}C_n x^{2n-n}\left(\frac{1}{x}\right)^n = {}^{2n}C_n\, x^n \frac{1}{x^n} = {}^{2n}C_n$$

$$= \frac{(2n)!}{n!\,n!} = \frac{2n(2n-1)(2n-3)......4.3.2.1}{n!\,n!}$$

$$= \frac{[1.3.5......(2n-1)\,2^n\, n\,(n-1)\,(n-2)\,(n-3).....2.1]}{n!\,n!}$$

$$= \frac{1.3.5....(2n-1)\,2^n n!}{n!\,n!} = \frac{1.3.5......(2n-1)2^n}{n!}$$

5 BINOMIAL COEFFICIENTS

In the binomial expansion of $(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$

- The binomial coefficients of the expansion equidistant from the beginning and the end are equal. In other words ${}^nC_r = {}^nC_{n-r}$.
- The greatest binomial coefficient in the expansion is always the binomial coefficient of middle term/terms.

Illustration 7

Question: **If the coefficient of $(2r + 4)^{th}$ term and $(r - 2)^{th}$ term in the expansion of $(1 + x)^{18}$ are equal, find r.**

Solution: Since coefficient of $(2r + 4)^{th}$ term in $(1 + x)^{18} = {}^{18}C_{2r+3}$.

Coefficient of $(r - 2)^{th}$ term $= {}^{18}C_{r-3}$

$\Rightarrow \quad {}^{18}C_{2r+3} = {}^{18}C_{r-3} \quad \Rightarrow \quad 2r + 3 + r - 3 = 18$

$\Rightarrow \quad 3r = 18 \quad \Rightarrow \quad r = 6.$

Illustration 8

Question: **Find the coefficient of x^5 in the expansion of $(1 + x + x^3)^9$.**

Solution: $(1+x+x^3)^9 = [(1+x)+x^3]^9$

$= (1+x)^9 + {}^9C_1\,(1+x)^8\,x^3 + {}^9C_2\,(1+x)^7\,x^6 + ... \qquad ...(i)$

The coefficient of x^5 in $(1+x)^9$ *is* ${}^9C_5$ i.e. ${}^9C_4$

The coefficient of x^5 in ${}^9C_1\,(1+x)^8\,x^3$ = coefficient of x^2 in ${}^9C_1\,(1+x)^8 = 9.\,{}^8C_2$

The remaining terms in (i) all contain powers of x higher than the fifth.

$\therefore$ the required coefficient is ${}^9C_4 + 9 \times {}^8C_2 = \frac{9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4} + \frac{9\cdot8\cdot7}{1\cdot2} = 378$

Alternatively

$$[1+x(1+x^2)]^9 = 1 + {}^9C_1\, x(1+x^2) + {}^9C_2\, x^2 (1+x^2)^2 + {}^9C_3\, x^3 (1+x^2)^3 + {}^9C_4\, x^4 (1+x^2)^4 + \ldots$$

x^5 occurs in 4th and 6th terms only and it is equal to $= 3 \times {}^9C_3 + {}^9C_5$

= 252 + 126 = 378

Illustration 9

Question: **Show that $11^{n+2} + 12^{2n+1}$ is divisible by 133.**

Solution: $11^{n+2} + 12^{2n+1} = 11^2 \cdot 11^n + 12\,(144)^n$

Now 144 and 121 should be expressed in terms of 133; 144 as (133 + 11) or 121 as (133 – 12)

$= 121 \cdot 11^n + 12\,(11+133)^n$

$= 11^n$ (121 + 12) + terms containing 133 as a factor

$= 11^n \cdot 133$ + terms containing 133 as a factor

Hence the expression is divisible by 133.

- *The ratio of (r + 1)th coefficient to r th coefficient is* $\dfrac{n-r+1}{r}$ *as*

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{\dfrac{n!}{(n-r)!\,r!}}{\dfrac{n!}{(n-r+1)!\,(r-1)!}} = \frac{n-r+1}{r}$$

Illustration 10

Question: **If a_r be the coefficient of x^r in the expansion of $(1 + x)^n$, then prove that**

$$(a_0 + a_1)(a_1 + a_2)\ldots(a_{n-1} + a_n) = a_0 \cdot a_1 \cdot a_2 \ldots a_{n-1} \frac{(n+1)^n}{n!}$$

Solution: $a_r = {}^nC_r$ and $a_{r-1} = {}^nC_{r-1}$

$$\therefore \quad \frac{a_r}{a_{r-1}} = \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n!}{r!\,(n-r)!} \cdot \frac{(r-1)!\,(n-r+1)!}{n!} = \frac{n-r+1}{r} = \frac{n+1}{r} - 1$$

$$\therefore \quad \frac{a_r + a_{r-1}}{a_{r-1}} = \frac{n+1}{r} \qquad \ldots(i)$$

Putting r = 1, 2, ….. n and multiplying together,

$$\left(\frac{a_0 + a_1}{a_0}\right)\left(\frac{a_1 + a_2}{a_1}\right)\ldots\left(\frac{a_{n-1} + a_n}{a_{n-1}}\right) = \frac{(n+1)^n}{1.2 \ldots n}$$

$$\therefore \quad (a_0 + a_1)(a_1 + a_2)\ldots(a_{n-1} + a_n) = \frac{(a_0\, a_1 \ldots a_{n-1})(n+1)^n}{n!}$$

6 GREATEST TERM IN THE EXPANSION

The numerically greatest term (absolute value) in the expansion of $(1 + x)^n$ is determined by the following process:

Let t_{r+1} be the numerically greatest term, then $|t_r| \le |t_{r+1}| \ge |t_{r+2}|$

Considering $\left|\dfrac{t_{r+1}}{t_r}\right| \ge 1$

$$\Rightarrow \quad \left|\frac{n-r+1}{r}x\right| \ge 1 \qquad \Rightarrow \quad \frac{n-r+1}{r}\,|x| \ge 1$$

$\Rightarrow \quad (n-r+1)|x| \ge r \quad \Rightarrow \quad r \le (n+1)\dfrac{|x|}{1+|x|}$ …(i)

Also $\left|\dfrac{t_{r+1}}{t_{r+2}}\right| \ge 1$

$\Rightarrow \quad \left|\dfrac{r+1}{n-r}\dfrac{1}{x}\right| \ge 1 \quad \Rightarrow \quad r+1 \ge (n-r)|x|$

$\Rightarrow \quad r \ge \dfrac{n|x|-1}{1+|x|}$

$\Rightarrow \quad r \ge (n+1)\dfrac{|x|}{1+|x|} - 1$ …(ii)

Thus from (i) and (ii) for t_{r+1} to be the greatest term,

$r \le m$ and $r \ge m-1$ where $m = (n+1)\dfrac{|x|}{1+|x|}$

- ***If m is an integer, then there are two terms with numerically greatest magnitude, for r = m and r = m – 1 i.e. t_{m+1} and t_m.***
- ***If m is not an integer, then there is only one numerically greatest term, for r = [m] i.e. $t_{[m]+1}$, where [m] is the greatest integer function of m.***

In general, for the expansion $(a + x)^n$ or $a^n\left(1+\dfrac{x}{a}\right)^n$ we can consider $m = (n+1)\dfrac{|x/a|}{1+|x/a|}$.

Illustration 11

Question: **Find the greatest term in the expansion of $(4 + 3x)^7$ when $x = \dfrac{2}{3}$.**

Solution: Here the greatest term means the numerically greatest term.

$$\left|\frac{t_{r+1}}{t_r}\right| = \frac{{}^7C_r\, 4^{7-r}\,(3x)^r}{{}^7C_{r-1}\, 4^{8-r}\,(3x)^{r-1}} = \frac{8-r}{r}\,\frac{3x}{4} = \frac{8-r}{2r} \text{ since } x = 2/3$$

Now $|t_{r+1}| \ge |t_r|$ if $8-r \ge 2r$ or $\dfrac{8}{3} \ge r$

This inequality is valid only for r = 1 or 2

Thus for r = 1, 2; $|t_{r+1}| > |t_r|$ and

for $r = 3, 4$; $|t_{r+1}| < |t_r|$

$\therefore \quad |t_1| < |t_2| < |t_3| > |t_4| > |t_5| > \ldots\ldots$

greatest term = $|t_3| = {}^7C_2\, 4^5 \cdot (3x)^2$, where $x = \dfrac{2}{3}$

$= 21 \times 4^5 \times 2^2 = 86016$

7 SUMMATION OF SERIES INCLUDING BINOMIAL COEFFICIENTS

In the binomial expansion of $(1 + x)^n$, let us denote the coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots,$ ${}^nC_r, \ldots\ldots, {}^nC_n$ by $C_0, C_1, C_2, \ldots\ldots, C_r, \ldots\ldots, C_n$ respectively.

- ***The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is 2^n :***

$\because \quad (1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_n x^n$

Putting $x = 1$

$\therefore \quad 2^n = C_0 + C_1 + C_2 + \ldots\ldots + C_n$ …(i)

or $\quad \displaystyle\sum_{r=0}^{n} C_r = 2^n$.

Illustration 12

Question: **Prove that the sum of the coefficients in the expansion of $(1 + x - 3x^2)^{2163}$ is –1.**

Solution: Putting $x = 1$ in $(1 + x - 3x^2)^{2163}$, the required sum of coefficients

$= (1 + 1 - 3)^{2163} = (-1)^{2163} = -1.$

Illustration 13

Question: **If the sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$ is equal to the sum of the coefficients in the expansion of $(x - \alpha y)^{35}$, then find the value of α.**

Solution: Sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$

= Sum of the coefficients in the expansion of $(x - \alpha y)^{35}$

Putting $x = y = 1$

$\therefore \quad (\alpha - 1)^{35} = (1 - \alpha)^{35}$

$\Rightarrow \quad (\alpha - 1)^{35} = -(\alpha - 1)^{35}$

$\Rightarrow \quad 2(\alpha - 1)^{35} = 0$

$\therefore \quad \alpha - 1 = 0$

$\therefore \quad \alpha = 1$

- ***The sum of the coefficients of the odd terms in the expansion of $(1 + x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to 2^{n-1}***

Since $(1 + x)^n = C_0 + C_1 + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$

Putting $x = -1$,

$0 = C_0 - C_1 + C_2 - C_3 + \ldots\ldots + (-1)^n C_n$

and $2^n = C_0 + C_1 + C_2 + C_3 + \ldots\ldots + C_n$ {from (i)}

Adding and subtracting these two equations, we get

$2^n = 2(C_0 + C_2 + C_4 + \ldots\ldots)$ and $2^n = 2(C_1 + C_3 + C_5 + \ldots\ldots)$

$\therefore \quad C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

sum of coefficients of odd terms = sum of coefficients of even terms = 2^{n-1}

Illustration 14

Question: **Evaluate the sum : ${}^8C_1 + {}^8C_3 + {}^8C_5 + {}^8C_7$.**

Solution: Since ${}^8C_1 + {}^8C_3 + {}^8C_5 + {}^8C_7$ = sum of even terms coefficients in the expansion of $(1 + x)^8$

$= 2^{8-1} = 2^7 = 128.$

- ***Differentiation can be used to solve series in which each term is a product of an integer and a binomial coefficient i.e. in the form $k \cdot {}^nC_r$.***

Illustration 15

Question: **Show that $3 \cdot C_0 + 7 \cdot C_1 + 11 \cdot C_2 + \ldots + (4n + 3)\, C_n = (2n + 3)\, 2^n$.**

Solution: This problem can be done be differentiating the expansion of $x^3(1 + x^4)^n$ and putting $x = 1$.

$x^3(1 + x^4)^n = x^3(C_0 + C_1x^4 + C_2x^8 + \ldots + C_nx^{4n})$

$= C_0x^3 + C_1x^7 + C_2x^{11} + \ldots + C_nx^{4n+3}$

Differentiating we get,

$\Rightarrow \quad 3x^2(1 + x^4)^n + x^3n(1 + x^4)^{n-1}4x^3 = 3x^2C_0 + 7x^6C_1 + 11x^{10}C_2 + \ldots + (4n + 3)x^{4n+2}C_n$

Now substituting $x = 1$ in both sides.

$\Rightarrow \quad 3C_0 + 7C_1 + 11C_2 + \ldots + (4n + 3)\, C_n$

$= 3(2^n) + 4n(2)^{n-1}$

$= (3 + 2n)\, 2^n$

Alternative method

Let $S = 3 \cdot C_0 + 7 \cdot C_1 + 11 \cdot C_2 + \ldots\ldots + (4n - 1)\, {}^nC_{n-1} + (4n + 3)\, C_n$

$S = (4n + 3)\, C_0 + (4n - 1)\, C_1 + \ldots + 3\, C_0$

Adding

$2S = (4n + 6)\{C_0 + C_1 + \ldots\ldots + C_n\} = (4n + 6)\, 2^n$

$\therefore \quad S = (2n + 3)\, 2^n$

Illustration 16

Question: **Show that $C_1 + 2 . C_2 + 3 . C_3 + ...+ n . C_n = n . 2^{n-1}$.**

Solution: The problem can be done by differentiating the expansion of $(1+x)^n$ and then putting $x=1$

Alternative method

The numbers multiplying binomial coefficients are 1, 2, 3, …., n and these are in arithmetic progression.

Let $S = C_1 + 2\cdot C_2 + 3\cdot C_3 + ... + (n-1)\,C_{n-1} + n\cdot C_n$

$$S = n\cdot C_0 + (n-1)\,C_1 + (n-2)\,C_2 + (n-3)\cdot C_3 + ... + 1\cdot C_{n-1}$$

(Writing the terms in the reverse order and remembering that $C_r = C_{n-r}$), adding

$$2S = n\cdot C_0 + n\,.\,C_1 + n\cdot C_2 + ... + n\cdot C_{n-1} + n\cdot C_n$$

$$= n\cdot[C_0 + C_1 + C_2 + ... C_n] = n\cdot 2^n$$

$\therefore \quad S = n\cdot 2^{n-1}$

Alternative method

$$S = {}^nC_1 + 2\cdot {}^nC_2 + 3\cdot {}^nC_3 + ... + n\cdot {}^nC_n$$

$$= n + \frac{2\cdot n\,(n-1)}{1.2} + 3\cdot\frac{n(n-1)\,(n-2)}{1\cdot 2\cdot 3} + ... + n$$

$$= n\left\{1 + \frac{n-1}{1} + \frac{(n-1)\,(n-2)}{1.2} + \frac{(n-1)(n-2)\,(n-3)}{1\cdot 2\cdot 3} + ... + 1\right\}$$

$$= n\left\{{}^{(n-1)}C_0 + {}^{(n-1)}C_1 + {}^{(n-1)}C_2 + ... + {}^{(n-1)}C_{n-1}\right\}$$

$$= n\cdot(1+1)^{n-1} = n\cdot 2^{n-1}$$

Illustration 17

Question: **Show that $C_1 - 2 . C_2 + 3 . C_3 - 4 . C_4 + ...+ (-1)^{n-1}\, n . C_n = 0$.**

Solution: The problem can be done by differentiating the expansion of $(1+x)^n$ and then putting $x = -1$.

Alternative method

L.H.S. $= {}^nC_1 - 2\cdot {}^nC_2 + 3\cdot {}^nC_3 - .... + (-1)^{n-1}\cdot n\cdot {}^nC_n$

$$= n\left\{1 - {}^{(n-1)}C_1 + {}^{(n-1)}C_2 - {}^{n-1}C_3 + ... + (-1)^{n-1}\,{}^{(n-1)}C_{n-1}\right\}$$

$$= n\,(1-1)^{n-1} = n \times 0 = 0$$

- *Integration can be used to solve series in which each term is a binomial coefficient divided by an integer i.e. in the form $\frac{{}^nC_r}{k}$.*

Illustration 18

Question: **Show that $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$**

Solution: Integrating the expansion of $(1 + x)^n$ between the limits 0 to 1.

$$\int_0^1 (1+x)^n\,dx = \int_0^1 (C_0 + C_1x + ... + C_nx^n)\,dx$$

$$\Rightarrow \quad \left.\frac{(1+x)^{n+1}}{n+1}\right|_0^1 = \left. C_0x + C_1\frac{x^2}{2} + ... + C_n\frac{x^{n+1}}{n+1}\right|_0^1$$

$$\Rightarrow \quad C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

Alternative method

L.H.S. $= 1 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \frac{{}^nC_3}{4} + ... + \frac{{}^nC_n}{n+1}$

$$= 1 + \frac{n}{2} + \frac{n(n-1)}{1\cdot 2\cdot 3} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} + \dots + \frac{n(n-1)(n-2)\dots 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\dots n\cdot(n+1)}$$

$$= \frac{1}{n+1}\left\{(n+1) + \frac{(n+1)n}{1\cdot 2} + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3} + \dots + \frac{(n+1)n(n-1)\dots 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\dots(n+1)}\right\}$$

$$= \frac{1}{(n+1)}\left\{{}^{(n+1)}C_1 + {}^{(n+1)}C_2 + {}^{(n+1)}C_3 + \dots + {}^{(n+1)}C_{n+1}\right\}$$

$$= \frac{1}{(n+1)}\{2^{n+1} - 1\} \text{ (from the expansion of } (1+1)^{n+1})$$

Illustration 19

Question: **Show that**

$$2\cdot C_0 + 2^2\cdot\frac{C_1}{2} + 2^3\cdot\frac{C_2}{3} + \dots + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{(n+1)} - 1}{n+1}$$

Solution: Integrating the expansion of $(1 + x)^n$ between the limits 0 to 2.

$$\int_0^2 (1+x)^n\,dx = \int_0^2 (C_0 + C_1 x + \dots + C_n x^n)\,dx$$

$$\Rightarrow \quad \left.\frac{(1+x)^{n+1}}{n+1}\right|_0^2 = \left. C_0 x + C_1\frac{x^2}{2} + \dots + C_n\frac{x^{n+1}}{n+1}\right|_0^2$$

$$\Rightarrow \quad 2\cdot C_0 + 2^2\cdot\frac{C_1}{2} + 2^3\cdot\frac{C_2}{3} + \dots + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{(n+1)} - 1}{n+1}$$

Alternative method

$$\text{L.H.S.} = \frac{1}{n+1}\left\{{}^{(n+1)}C_1\cdot 2 + {}^{(n+1)}C_2\cdot 2^2 + {}^{(n+1)}C_3\cdot 2^3 + \dots + {}^{(n+1)}C_{n+1}\cdot 2^{(n+1)}\right\}$$

$$= \frac{1}{n+1}\left\{1 + {}^{(n+1)}C_1\cdot 2 + {}^{(n+1)}C_2\cdot 2^2 + \dots + {}^{(n+1)}C_{n+1}\cdot 2^{(n+1)} - 1\right\}$$

$$= \frac{1}{(n+1)}\left\{(1+2)^{n+1} - 1\right\} = \frac{3^{n+1} - 1}{n+1}$$

- ***Product of two expansions can be used to solve some problems related to series of binomial coefficients in which each term is a product of two binomial coefficients.***

Illustration 20

Question: **Show that $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = \frac{(2n)!}{n!\,n!}$**

Solution: This example can be solved by considering two binomial expansions $(1+x)^n$ and $\left(1+\frac{1}{x}\right)^n$ in which the coefficients of x^n and $\frac{1}{x^n}$ are equal and in the product of these expansions the constant term will contain the square of binomial coefficients.

Consider $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_n x^n$

$$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots + \frac{C_n}{x^n}$$

Taking the product of these two expansions and collecting the constant term in the product, Constant term in R.H.S. $= C_0^2 + C_1^2 + C_2^2 + ... C_n^2$

$= $ constant term in L.H.S. = constant term in $(1+x)^n \left(1+\frac{1}{x}\right)^n$

$= $ constant term in $\frac{(1+x)^{2n}}{x^n} =$ coefficient of x^n in $(1+x)^{2n}$

$$= {}^{2n}C_n = \frac{(2n)!}{(n!)\,(n!)}$$

Illustration 21

Question: **Show that** $C_0^2 - C_1^2 + C_2^2 - C_3^2 + ... + (-1)^n C_n^2$

$$= \begin{cases} 0, \text{ if } n \text{ is odd} \\ \dfrac{(-1)^{n/2}\, n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}, \text{ if } n \text{ is even.} \end{cases}$$

Solution: Consider the product of the expansion of $(1+x)^n$ and $\left(1-\frac{1}{x}\right)^n$ and compare the constant term.

$C_0^2 - C_1^2 + C_2^2 + ... + (-1)^n C_n^2 =$ constant term in $(1+x)^n \left(1-\frac{1}{x}\right)^n$

$=$ constant term in $\frac{(1+x)^n\,(x-1)^n}{x^n}$

$=$ constant term in $\frac{(-1)^n (1-x^2)^n}{x^n}$

$=$ coefficient of x^n in $(-1)^n\,(1-x^2)^n$

$= 0$, if n is odd since all the terms in $(1-x^2)^n$ contain only even power of x

$=$ coefficient of x^{2m} in $(-1)^{2m}(1-x^2)^{2m}$, if n is even = 2m

$$= (-1)^m\ {}^{2m}C_m = (-1)^m\ {}^{2m}C_m = (-1)^m \frac{(2m)!}{m!\,m!} = \frac{n!\,(-1)^{n/2}}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$$

Illustration 22

Question: **Show that**

$$C_0\,C_r + C_1\,C_{r+1} + C_2\,C_{r+2} + ... + C_{n-r}\,C_n = \frac{(2n)!}{(n-r)!\,(n+r)!}$$

Solution: Consider $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_r x^r + C_{r+1} x^{r+1} + ... + C_n x^n$

and $\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + ... + \frac{C_r}{x^r} + \frac{C_{r+1}}{x^{r+1}} + ... + \frac{C_n}{x^n}$

In the product of these two expansions, collecting the coefficient of x^r

$C_0\,C_r + C_1\,C_{r+1} + C_2\,C_{r+2} + ... + C_{n-r}\,C_n =$ coefficient of x^r in $\frac{(1+x)^{2n}}{x^n}$

$=$ coefficient of x^{n+r} *in* $(1+x)^{2n}$

$$= {}^{2n}C_{n+r} = \frac{(2n)!}{(n+r)!\,(n-r)!}$$

8 IMPORTANT RESULTS

8.1 *If* $(\sqrt{P}+Q)^n = I+f$ *where I and n are positive integers, n being odd, and* $0 \le f < 1$*, then show that* $(I+f)f = k^n$ *where* $P - Q^2 = k > 0$ *and* $\sqrt{P} - Q < 1$.

Proof. Given $\sqrt{P} - Q < 1$

$\therefore \quad 0 < (\sqrt{P}-Q)^n < 1$

Now let $(\sqrt{P}-Q)^n = f'$ where $0 < f' < 1$

$\therefore \quad I + f - f' = (\sqrt{P}+Q)^n - (\sqrt{P}-Q)^n$

$\because$ RHS contains even powers of $\sqrt{P}$ ($\because$ n is odd)

$\Rightarrow$ RHS is an integer

Since RHS and I are integers,

$\therefore$ f – f′ is also integer.

$\therefore \Rightarrow$ f – f′ = 0 $\because -1 < f - f' < 1$

or f = f′

$\therefore \quad (I+f)f = (I+f)f' = (\sqrt{P}+Q)^n(\sqrt{P}-Q)^n = (P-Q^2)^n = k^n$.

8.2 *If* $(\sqrt{P}+Q)^n = I+f$ *where I and n are positive integer, n being even, and* $0 \le f < 1$*, then show that* $(I+f)(1-f) = k^n$ *where* $P - Q^2 = k > 0$ *and* $\sqrt{P} - Q < 1$.

Proof : If n is an even integer then

$$(\sqrt{P}+Q)^n + (\sqrt{P}-Q)^n = I + f + f'$$

Hence LHS and I are integer.

$\therefore$ f + f′ is also integer.

$\Rightarrow$ f + f′ = 1 $\because 0 < f + f' < 2$

$\therefore$ f′ = (1 – f)

Hence $(I+f)(1-f) = (I+f)f' = (\sqrt{P}+Q)^n(\sqrt{P}-Q)^n$

$= (P - Q^2)^n = k^n$.

Illustration 23

Question: **If $(2+\sqrt{3})^n = I + f$ where I and n are positive integers and $0 < f < 1$, show that (i) I is an odd integer and (ii) $(I+f)(1-f) = 1$.**

Solution: (i) Now $0 < 2 - \sqrt{3} < 1$, since $2 - \sqrt{3} = 0.268$ (approx.)

$\therefore 0 < (2-\sqrt{3})^n < 1$; we can take $(2-\sqrt{3})^n$ as f'.

Now $(2+\sqrt{3})^n + (2-\sqrt{3})^n = I + f + f'$

But L.H.S. $= 2\left\{2^n + {}^nC_2 2^{n-2}(\sqrt{3})^2 + {}^nC_4 2^{n-4}(\sqrt{3})^4 + ...\right\}$ = an integer

(in fact an even integer)

$\therefore$ RHS. = I + f + f′ = an even integer

Also $f + f' = 1$, since f and f ’ are both positive proper fractions.

$\therefore$ I = an even integer –1 = an odd integer.

(ii) $(I+f)(1-f) = (I+f)(f') = (2+\sqrt{3})^n \cdot (2-\sqrt{3})^n$

$= (4-3)^n = 1^n = 1$

Illustration 24

Question: **Let $R = (5\sqrt{5}+11)^{2n+1}$ and $f = R - [R]$, where [ ] denotes the greatest integer function. Prove that $Rf = 4^{2n+1}$.**

Solution: Greatest integer function is defined as follows:

$[x]$ = greatest integer $\leq x$

In the case of positive number, x

$[x]$ =integral part of x

$\therefore f = R - [R]$ implies that f is the fractional part of R

$\therefore 0 < f < 1$

Since $144 > 125 > 121, \sqrt{125} = 5\sqrt{5}$ lies between 11 and 12.

$\therefore 0 < 5\sqrt{5} - 11 < 1$ and hence $(5\sqrt{5} - 11)^{2n+1}$ will also be a proper fraction.

Let $g = (5\sqrt{5} - 11)^{2n+1}$

Now $[R] + f - g = R - g$

$= (5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1}$

$= 2\left\{ {}^{(2n+1)}C_1 \, (5\sqrt{5})^{2n} \cdot 11^1 + {}^{(2n+1)}C_3 \, (5\sqrt{5})^{2n-2} \cdot 11^2 + ... \right\}$

$=$ an even integer

Since $[R]$ is an integer, the above implies $f - g = 0$ (*i.e.*) $f = g$.

Hence $Rf = Rg = (5\sqrt{5} + 11)^{2n+1} \cdot (5\sqrt{5} - 11)^{2n+1}$

$= (125 - 121)^{2n+1} = 4^{2n+1}$

9 MULTINOMIAL EXPANSION

If $n \in N$, then the general term of the multinomial expansion $(x_1 + x_2 + x_3 + ... + x_k)^n$ is

$$\frac{n!}{a_1!\, a_2!\, a_3! ..... a_k!} x_1^{a_1} \cdot x_2^{a_2} \cdot x_3^{a_3} ...... x_k^{a_k} ,$$

where $a_1 + a_2 + a_3 + ... + a_k = n$

and $0 \leq a_i \leq n,\ i = 1, 2, 3, ... k$.

and the number of terms in the expansion are ${}^{n+k-1}C_{k-1}$.

Illustration 25

Question: **Find the total number of terms in the expansion of $(x + y + z + w)^n$, $n \in N$.**

Solution: The number of terms in the expansion of $(x + y + z + w)^n$ is ${}^{n+4-1}C_{4-1}$.

$= {}^{n+3}C_3$

$= \dfrac{(n+3)(n+2)(n+1)}{6}$

Alternative method

We know that

$$(x + y + z + w)^n = \{(x + y) + (z + w)\}^n$$

$$= (x + y)^n + {}^nC_1((x + y)^{n-1}(z + w) + {}^nC_2(x + y)^{n-2}(z + w)^2 + ... + {}^nC_n(z + w)^n$$

$\therefore$ Number of terms in RHS.

$$= (n+1) + n.2 + (n-1).3 + ... + 1.(n+1)$$

$$= \sum_{r=0}^{n} (n - r + 1)(r + 1)$$

$$= \sum_{r=0}^{n} \left((n+1) + nr - r^2\right)$$

$$= (n+1)\sum_{r=0}^{n} 1 + n\sum_{r=0}^{n} r - \sum_{r=0}^{n} r^2$$

$$= (n+1).(n) + n.\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(n+2)(n+3)}{6}$$

10 BINOMIAL THEOREM FOR ANY INDEX

If $n \in R$, $-1 < x < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + \frac{n(n-1)(n-2).....(n-r+1)}{r!}x^r + ...\infty$$

Note:

(i) In the expansion if $n \in \boldsymbol{R}$ and $n > 0$ then $-1 < x < 1$.

(ii) ^nC_r can not be used because it is defined only for natural number.

(iii) If x be so small then its square and higher powers may be neglected, then approximate value of $(1 + x)^n = 1 + nx$.

Important results:

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + ... + (-1)^r x^r + ...$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...$
- $(1+x)^{-2} = 1 - {}^2C_1x + {}^3C_2x^2 - {}^4C_3x^3 + ... + (-1)^r\ {}^{r+1}C_rx^r + ...$
- $(1-x)^{-2} = 1 + {}^2C_1x + {}^3C_2x^2 + {}^4C_3x^3 + ... + {}^{r+1}C_rx^r + ...$
- $(1+x)^{-3} = 1 - {}^3C_1x + {}^4C_2x^2 - {}^5C_3x^3 + ... + (-1)^r\ {}^{r+2}C_rx^r + ...$
- $(1-x)^{-3} = 1 + {}^3C_1x + {}^4C_2x^2 + {}^5C_3x^3 + ... + {}^{r+2}C_rx^r + ...$

In general, the coefficient of x^n in $(1 - x)^{-k}$ is ${}^{n+k-1}C_{k-1}$