MATHS BOOKLET FOR JEE (MAINS & ADVANCE) & BOARDS

Complex Number

1 INTRODUCTION

Whenever $\sqrt{x}$ is thought of to give a real value, it has been, till now, insisted that $x \geq 0$. In other words, in the set of real numbers it is not possible to provide for the existence of a value for $\sqrt{x}$ when $x < 0$. To make this possible we extend the number system so as to include and cover yet another class of numbers, called imaginary numbers.

Let us take the quadratic equation $x^2 - 2x + 10 = 0$. The formal solution of this equation is $\frac{2 \pm \sqrt{4-40}}{2}$ i.e., $1 \pm 3\sqrt{-1}$, which is not meaningful in the set of real numbers.

It is therefore, the symbol i, is thought of to possess the following properties:

(i) It combines with itself and with real numbers satisfying the laws of algebra.

(ii) Whenever we come across –1 we may substitute i^2.

In the light of the foregoing the roots of the equation discussed earlier may be taken as $1 + 3i$, $1 - 3i$.

It is taken that 1 is real part and 3(or – 3) is the imaginary part of this complex number $1 + 3i$ or $1 - 3i$ respectively.

It has now to be mentioned that, + symbol standing between 1 and $3i$ does not appear to be meaningful; though the following are true.

$$(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2) \quad \text{.... (i)}$$

i.e., the real parts are added (or subtracted) separately and so in fact the imaginary parts.

Also,
$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \quad \text{.... (ii)}$$

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \quad \text{.... (iii)}$$

To make these operations really meaningful, a formal extension of the number system is done in this lesson.

2 COMPLEX NUMBERS

A complex number, represented by an expression of the form $x + iy$ (x, y are real), is taken to be an ordered pair (x, y) of two real numbers, combined to form a complex number and an algebra is defined on the set of such numbers, represented by an ordered pair (x, y) to satisfy the following:

(addition) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

(subtraction) $(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$

(multiplication) $(x_1, y_1) \times (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

(division) $(x_1, y_1) \div (x_2, y_2) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{x_2y_1 + x_1y_2}{x_2^2 + y_2^2}\right)$

For any real number $\alpha, \alpha(x, y) = (\alpha x, \alpha y)$ and if $(x, y) = (x', y')$ then it must be $x' = x$; $y' = y$. In other words, the representation of a complex number in the form (x, y) has a uniqueness property; and for a complex number it is not possible to have two different ordered pairs form of representation. In the light of the foregoing it may be said that the two representation (x, y) – in the ordered pair form and $x + iy$ are indistinguishable.

Based on the above discussion we are listing a few points:

1. If $z = a + ib$, then real part of z = Re $(z) = a$ and Imaginary part of $z = Im(z) = b$.
2. If Re $(z) = 0$, the complex number is purely imaginary.
3. If Im $(z) = 0$, the complex number is real.
4. The complex number $0 = 0 + 0i$ is both purely imaginary and real.
5. Two complex numbers are equal if and only if their real parts and imaginary parts are separately equal i.e. $a + ib = c + id \Leftrightarrow a = c$ and $b = d$.
6. There is no order relation between complex numbers i.e. $(a + ib) >$ or $< (c + id)$ is a meaningless expression.

3 REPRESENTATION OF A COMPLEX NUMBER

3.1 GEOMETRICAL REPRESENTATION

It is known, from coordinate geometry, that the ordered pair (x, y) represents a point in the Cartesian plane.

It is now seen that the ordered pair (x, y) taken as Z represents a complex number.

It is therefore, that to every complex number $Z \equiv (x, y)$, one can associate, a point $P \equiv (x, y)$ in the Cartesian plane. The point may be said to be geometrical representation of Z. This association is a bijection – in the mapping language – whereby, this correspondence between Z and P is ONE-ONE and ONTO. It is therefore possible to go over to a point from Z, or reversing the roles, come back to Z from the point.

3.2 ARGAND DIAGRAM

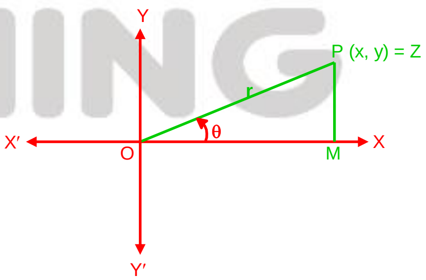
The graphical representation of a complex number $Z = (x, y)$ by a point $P(x, y)$ is called representation in the Argand's Diagram also called Gaussian plane. In this representation, all complex numbers like (2, 0), (3, 0), (–1, 0), $(\alpha, 0)$ with imaginary part 0 will be represented by points on the x-axis. Since the real number α is represented as a complex number $(\alpha, 0)$, all real numbers will get marked on the x-axis. For this reason, the x-axis is called the real axis. Similarly all purely imaginary numbers (with real part 0) like (0, 1), (0, 2), (0, – 3), $(0, \beta)$ will be marked on the y-axis. Hence the y-axis is also called the imaginary axis in this context. The Cartesian plane (two dimensional plane) is also called the complex plane.

3.3 POLAR REPRESENTATION

Let $P(x, y)$ be any point on the complex plane representing the complex number $z = (x, y)$, with $X'OX$ and $Y'OY$ as the axes of coordinates.

Let $OP = r$ and $\angle XOP = \theta$ (measured in anticlockwise).

Then from ΔOMP, we find that $x = OM = r\cos\theta$ and $y = MP = r\sin\theta$



Thus $z = (x, y) = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

where $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$ by eulers formula

Thus $z = r(\cos\theta + i\sin\theta)$ can be written as

$\boldsymbol{z = re^{i\theta}}$

This form of representation of Z is called the **trigonometric form** or the **polar form** or the **modulus amplitude form**.

When z is written in the form $r(\cos\theta + i\sin\theta)$, r is called the modulus of z and is written as $|z|$; $|z| = r = \sqrt{x^2+y^2}$, a non-negative number. $|z| = 0$ for the only number (0, 0).

3.4 VECTOR REPRESENTATION OF A COMPLEX NUMBER

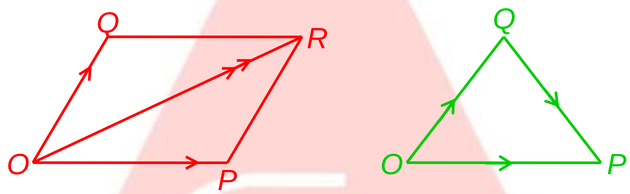
In the Argand's diagram any complex number $Z = x + iy$ can be represented by a point P with coordinates (x, y). The vector $\overline{OP}$ can also be used to represent Z. The length of the vector $\overline{OP}$, (i.e.,) OP is the modulus of Z and the angle θ that OP makes with the positive X-axis is the amplitude of Z.

Representation of an algebraic operation on complex numbers

Sum: If two complex numbers Z_1 and Z_2 be represented by the points P and Q or by $\overline{OP}$ and $\overline{OQ}$, then the sum $Z_1 + Z_2$ is represented by R or $\overline{OR}$, where $\overline{OR} = \overline{OP} + \overline{OQ}$ and OR is the diagonal of the parallelogram with OP and OQ as adjacent sides.

Difference: $Z_1 - Z_2$ will be represented by $\overline{QP}$ since $\overline{QP} = \overline{OP} - \overline{OQ}$.

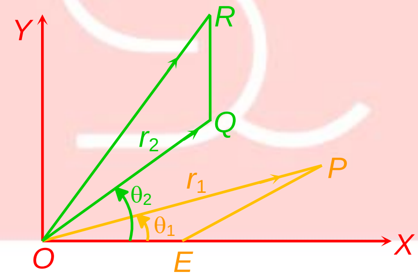
$Z_2 - Z_1$ will be represented by $\overline{PQ}$.



Multiplication: If $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

then $Z_1 Z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$.

If $\overline{OP}$ and $\overline{OQ}$ represent Z_1 and Z_2, construct ΔOQR similar to ΔOEP where $OE = 1$.



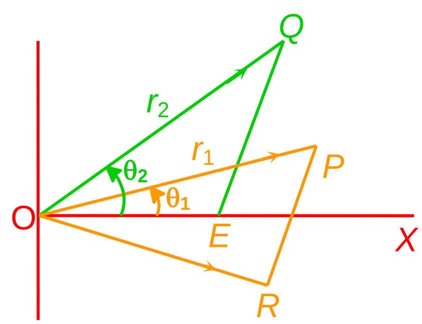
$$\lfloor XOR = \lfloor XOQ + \lfloor QOR = \lfloor XOQ + \lfloor EOP = \theta_2 + \theta_1$$

and $\dfrac{OR}{OQ} = \dfrac{OP}{OE}$, $\quad \therefore OR = OP \cdot OQ = r_1 r_2 \quad$ {as $OE = 1$}

Hence $\overline{OR}$ represents the product $Z_1 Z_2$.

Division

$$\frac{Z_1}{Z_2} = \left(\frac{r_1}{r_2}\right)\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$



Construct ΔORP similar to ΔOEQ

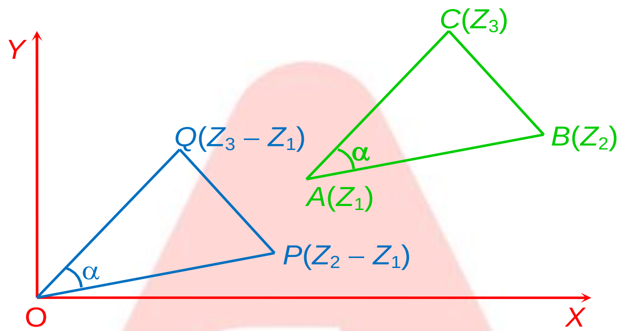
Now $\frac{OR}{OE} = \frac{OP}{OQ} \Rightarrow OR = \frac{r_1}{r_2}$ and $\angle ROX = \angle ROP - \angle EOP = \angle EOQ - \angle EOP = \theta_2 - \theta_1$

$\therefore \angle XOR = \theta_1 - \theta_2$

Hence $\overline{OR}$ represents $\frac{Z_1}{Z_2}$.

Corollary 1: If Z_1, Z_2, Z_3 are the vertices of a triangle *ABC* described in the counter-clockwise direction, then

$\frac{Z_3 - Z_1}{Z_2 - Z_1} = \frac{CA}{BA}$ (cos α + *i* sin α), where $\alpha = \angle BAC$



Let *P* and *Q* be the points representing $Z_2 - Z_1$ and $Z_3 - Z_1$.
Then the triangles *POQ* and *BAC* are congruent.

$\therefore \frac{CA}{BA} = \frac{OQ}{OP}$ and $\angle QOP = \angle BAC = \alpha$

Now $\frac{Z_3 - Z_1}{Z_2 - Z_1}$ has modulus $\frac{OQ}{OP} = \frac{CA}{BA}$ and argument $\angle POQ = \alpha$

Hence $\frac{Z_3 - Z_1}{Z_2 - Z_1} = \left(\frac{CA}{BA}\right)$ (cosα + *i* sinα)

In particular, if α = 90° and *AB* = *AC*, then $\frac{Z_3 - Z_1}{Z_2 - Z_1} = i$ or $(Z_3 - Z_1) = i\,(Z_2 - Z_1)$

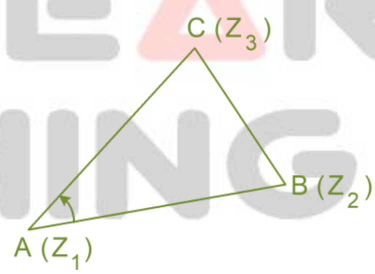
Corollary 2

If Z_1, Z_2, Z_3 are represented by *A*, *B*, *C*, then

$\arg\left(\frac{Z_3 - Z_1}{Z_2 - Z_1}\right) = \angle BAC$

$\arg\left(\frac{Z_2 - Z_3}{Z_1 - Z_3}\right) = \angle ACB$ and

$\arg\left(\frac{Z_1 - Z_2}{Z_3 - Z_2}\right) = \angle CBA$



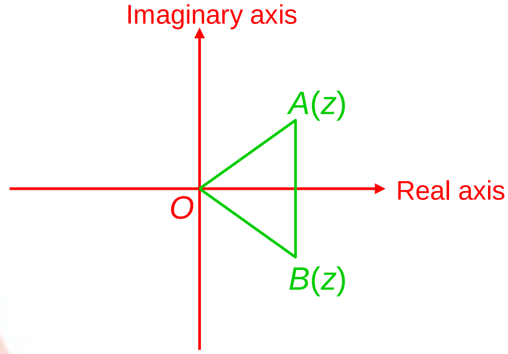
4 CONJUGATE OF A COMPLEX NUMBER

The complex numbers $z = (a, b) = a + ib$ and $\bar{z} = (a, -b) = a - ib$, where *a* and *b* are real numbers, $i = \sqrt{-1}$ and $b \neq 0$ are said to be complex conjugate of each other. (Here the complex conjugate is obtained by just changing the sign of *i*).

Note that, sum $= (a + ib) + (a - ib) = 2a$ which is real
and product $= (a + ib)(a - ib) = a^2 - (ib)^2 = a^2 - i^2b^2$
$= a^2 - (-1)\,b^2 = a^2 + b^2$ which is real.

4.1 PROPERTIES OF CONJUGATE

- $\overline{(\bar{z})} = z$
- $z = \bar{z} \iff z$ is real
- $z = -\bar{z} \iff z$ is purely imaginary
- $\text{Re}(z) = \text{Re}(\bar{z}) = \dfrac{z + \bar{z}}{2}$
- $\text{Im}(z) = \dfrac{z - \bar{z}}{2i}$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}} \quad (z_2 \neq 0)$
- $z_1\overline{z_2} + \overline{z_1}z_2 = 2\text{Re}(\overline{z_1}z_2) = 2\text{Re}(z_1\overline{z_2})$
- $\overline{z^n} = (\bar{z})^n$
- If $z = f(z_1)$, then $\bar{z} = f(\overline{z_1})$



5 MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = x + iy$ is a real number given by $|z| = \sqrt{x^2 + y^2}$. It is always non-negative and $|z| = 0$ only for $z = 0$ i.e. origin of Argand plane. Geometrically it represents the distance of the point complex number from its origin.

5.1 PROPERTIES OF MODULUS

- $|z| \geq 0 \Rightarrow |z| = 0$ iff $z = 0$ and $|z| > 0$ iff $z \neq 0$.
- $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq |z|$.
- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $z\bar{z} = |z|^2$
- $|z_1 z_2| = |z_1|\,|z_2|$

  In general $|z_1 z_2 z_3 \ldots\ldots z_n| = |z_1|\,|z_2|\,|z_3| \ldots\ldots |z_n|$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} \quad (z_2 \neq 0)$
- $|z_1 \pm z_2| \leq |z_1| + |z_2|$

  In particular, if $|z_1 + z_2| = |z_1| + |z_2|$, then origin, z_1 and z_2 are collinear with origin at one of the ends.
- $|z_1 \pm z_2| \geq \big|\,|z_1| - |z_2|\,\big|$

  In particular, if $|z_1 - z_2| = \big|\,|z_1| - |z_2|\,\big|$, then origin, z_1 and z_2 are collinear with origin at one of the ends.
- $|z^n| = |z|^n$
- $\big|\,|z_1| - |z_2|\,\big| \leq |z_1| + |z_2|$

  Thus $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $\big|\,|z_1| - |z_2|\,\big|$ is the least possible value of $|z_1 + z_2|$
- $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\overline{z_1} \pm \overline{z_2}) = |z_1|^2 + |z_2|^2 \pm (z_1\overline{z_2} + \overline{z_1}z_2)$ or

  $|z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1\overline{z_2})$
- $z_1\overline{z_2} + \overline{z_1}z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$ where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.

- $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.
- $|z_1+z_2|^2 + |z_1-z_2|^2 = 2\left\{|z_1|^2+|z_2|^2\right\}$
- $|az_1-bz_2|^2 + |bz_1+az_2|^2 = (a^2+b^2)\,(|z_1|^2+|z_2|^2)$ where $a, b \in \boldsymbol{R}$.
- Unimodular : i.e., unit modulus
  If z is unimodular then $|z| = 1$. A unimodular complex number can always be expressed as $\cos\theta + i\sin\theta$, $\theta \in \boldsymbol{R}$.

  Note: $\dfrac{z}{|z|}$ is always a unimodular complex number if $z \neq 0$.

Some of the proofs are given as:

- $|Z_1 Z_2| = |Z_1| \times |Z_2|$

Proof:

Let $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then $Z_1 Z_2 = r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\} = r(\cos\theta + i\sin\theta)$,

where $r = r_1 r_2$ and $\theta = \theta_1 + \theta_2$.

$\therefore |Z_1 Z_2| = r = r_1 r_2 = |Z_1| \times |Z_2|$

- $|Z_1 Z_2 \ldots Z_n| = |Z_1| \times |Z_2| \times |Z_3| \times \ldots \times |Z_n|$

  Proof follows by writing $Z_1 Z_2 \ldots Z_n$ as the product of $Z_1 Z_2 \ldots Z_{n-1}$ and Z_n and applying property (1) repeatedly.
- $|Z^n| = |Z|^n$

  Proof follows if we take $Z_1 = Z_2 = Z_3 = \ldots = Z_n$
- $\left|\dfrac{Z_1}{Z_2}\right| = \dfrac{|Z_1|}{|Z_2|}$

Proof:

Let $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Now $\dfrac{Z_1}{Z_2} = \dfrac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)} = \dfrac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)$

$$\left(\text{since } \frac{1}{\cos\theta_2 + i\sin\theta_2} = \cos\theta_2 - i\sin\theta_2\right)$$

$$= \left(\frac{r_1}{r_2}\right)\{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\}$$

$$= \left(\frac{r_1}{r_2}\right)\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$$

Hence $\left|\dfrac{Z_1}{Z_2}\right| = \dfrac{r_1}{r_2} = \dfrac{|Z_1|}{|Z_2|}$

- First triangle inequality $|Z_1| + |Z_2| \geq |Z_1 + Z_2|$

Proof:

$$|Z_1 + Z_2| = |r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)|$$

$$= |(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)|$$

$$= \sqrt{(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2} = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1-\theta_2)}$$

$\leq \sqrt{r_1^2 + r_2^2 + 2r_1r_2}$, since $\cos(\theta_1 - \theta_2) \leq 1$

$\therefore |Z_1 + Z_2| \leq \sqrt{(r_1+r_2)^2}$

or $|Z_1 + Z_2| \leq r_1 + r_2$. Thus $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$.

Note: Equality occurs only when $\theta_1 = \theta_2$ i.e. when Z_1 and Z_2 have the same amplitude.

- Second triangle inequality
  $|Z_1 - Z_2| \geq |Z_1| \sim |Z_2|$

Proof

$$Z_1 - Z_2 = r_1\cos\theta_1 - r_2\cos\theta_2 + i(r_1\sin\theta_1 - r_2\sin\theta_2)$$

$$\therefore |Z_1 - Z_2| = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2} = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

$$\geq \sqrt{r_1^2 + r_2^2 - 2r_1r_2}\text{, since } \cos(\theta_1 - \theta_2) \leq 1$$

$$\therefore |Z_1 - Z_2| \geq \sqrt{(r_1 - r_2)^2} = |r_1 - r_2|$$

$$|Z_1 - Z_2| \geq r_1 \sim r_2 = |Z_1| \sim |Z_2|.$$

- $|\bar{Z}| = |Z|$

Proof:

$|Z| = \sqrt{x^2 + y^2}$ if $Z = x + iy$

Then $\bar{Z} = x - iy$

$$\therefore |\bar{Z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2}$$

$$\therefore |\bar{Z}| = |Z|$$

6 ARGUMENT OF A COMPLEX NUMBERS

If $z = x + iy = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$.

θ is called the argument of Z or amplitude of Z. Since $x = r\cos\theta$ and $y = r\sin\theta$, θ is such that $\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$. Since there can be many values of θ satisfying these conditions, by convention, θ such that $-\pi < \theta \leq \pi$ is defined as the principal argument of Z and is denoted by arg Z. The argument of a complex number $a + ib$ is given by α, $\pi - \alpha$, $-\pi + \alpha$, or $-\alpha$ if $a + ib$ is in first, second, third or fourth quadrant respectively, where $\alpha = \tan^{-1}\left|\frac{b}{a}\right|$.

For example

- $Z = 1 + i = (1, 1)$ and is marked by point $P(1, 1)$ lies in first quadrant.
  $\therefore |Z| = \sqrt{2}$ and arg $Z = \pi/4$.
- If $Z = 1 - i = (1, -1)$, then P lies in the fourth quadrant and $|Z| = \sqrt{2}$ and arg $Z = -\pi/4$.
- If $Z = -1 + i = (-1, 1)$, then P lies in the second quadrant and arg $Z = \frac{3\pi}{4}$.
- If $Z = -1 - i$, arg $Z = -\frac{3\pi}{4}$.
- Argument of all positive real numbers like 1, 2, 3, $\frac{1}{2}$, … is 0 since they are marked on the positive x-axis. Argument of all negative real numbers like – 1, – 2, – 3, … is π since they are marked on OX′. Argument of purely imaginary numbers like i, $2i$, $3i$, … is $\frac{\pi}{2}$ since these are marked on the positive y-axis. Argument of purely imaginary numbers like $-i$, $-2i$, $-3i$, … is $-\frac{\pi}{2}$.

6.1 PROPERTIES OF ARGUMENTS

- Arg (z_1z_2) = Arg (z_1) + Arg (z_2) + $2k\pi$ ($k = 0$ or 1 or – 1)

In general Arg $(z_1z_2z_3 \ldots\ldots z_n)$ = Arg (z_1) + Arg (z_2) + Arg (z_3) + …… + Arg (z_n) + $2k\pi$ (where $k \in I$)

- Arg $\left(\frac{z_1}{z_2}\right)$ = Arg z_1 – Arg z_2 + $2k\pi$ ($k = 0$ or 1 or –1)
- Arg $\left(\frac{z}{\bar{z}}\right) = 2$ Arg $z + 2k\pi$ ($k = 0$ or 1 or –1)
- Arg $(z^n) = n$ Arg $z + 2k\pi$ ($k = 0$ or 1 or –1)
- If Arg $\left(\frac{z_2}{z_1}\right) = \theta$, then Arg $\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$ where $k \in I$.
- Arg $\bar{z}$ = – Arg z
- If arg $(z) = 0 \Rightarrow z$ is real.

Note: Proper value of k must be chosen so, that R.H.S. of (i), (ii), (iii), (iv) lies in $(-\pi, \pi]$.
All the above formulae are written on the basic of principal argument.

7 De MOIVRE'S THEOREM

For any rational number n, the value or one of the values of $(\cos\theta + \sin\theta)^n$ is $(\cos n\theta + \sin n\theta)$.
The following may also be noted:

(a) $(\cos\theta + i\sin\theta)^{-n} = (\cos n\theta - \sin n\theta) = (\cos\theta - i\sin\theta)^n$

(b) $(\cos\theta + i\sin\theta)^n = (\cos n\theta + \sin n\theta) = (\cos\theta - i\sin\theta)^{-n}$

(c) If $x + \frac{1}{x} = 2\cos\theta$; and if the equation is solved for x then $x = \cos\theta + i\sin\theta = e^{i\theta}$; then $\frac{1}{x} = \cos\theta - i\sin\theta = e^{-i\theta}$

or, $x = \cos\theta - i\sin\theta = e^{-i\theta}$; then $\frac{1}{x} = \cos\theta + i\sin\theta = e^{i\theta}$

8 ROOTS OF UNITY

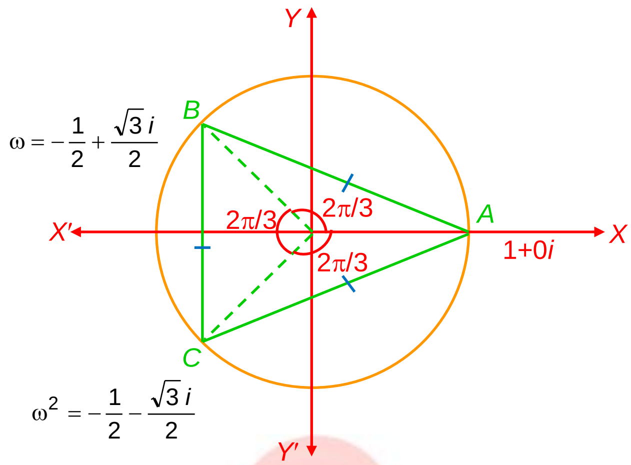
8.1 CUBE ROOTS OF UNITY

Consider the cubic (3rd degree) equation

$$x^3 = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

$$\therefore x = \sqrt[3]{1} = (\cos 2k\pi + i\sin 2k\pi)^{1/3}$$

$$= \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)$$



To get the three roots of the cubic equation we give

$k = 0$, giving the real root, $\cos 0 + i \sin 0 = 1$

$k = 1$, giving one imaginary root, $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = \omega$

$k = 2$, giving the other imaginary root, $\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = \omega^2$

It is said that 1, ω, ω^2 are the three cubic roots of unity satisfying

(i) $1 + \omega + \omega^2 = 0$

(ii) $\omega^3 = 1$

(iii) 1, ω, ω^2 are represented respectively by points A, B, C lying on the unit circle $|Z| = 1$ and forming the corners of an equilateral triangle of sides $\sqrt{3}$.

8.2 Some useful results

$(x^3 + y^3) = (x + y)(x + \omega y)(x + \omega^2 y)$

$(x^3 - y^3) = (x - y)(x - \omega y)(x - \omega^2 y)$

$(x^3 + y^3 + z^3 - 3xyz) = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

8.3 n^{th} ROOTS OF UNITY

More generally the nth degree equation $x^n = 1$ has 'n' nth roots of unity given by

$$\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}, \cos \frac{4\pi}{n} + i \sin \frac{4\pi}{n}, \ldots, \cos \frac{2(n-1)\pi}{n} + i \sin \frac{2(n-1)\pi}{n}$$

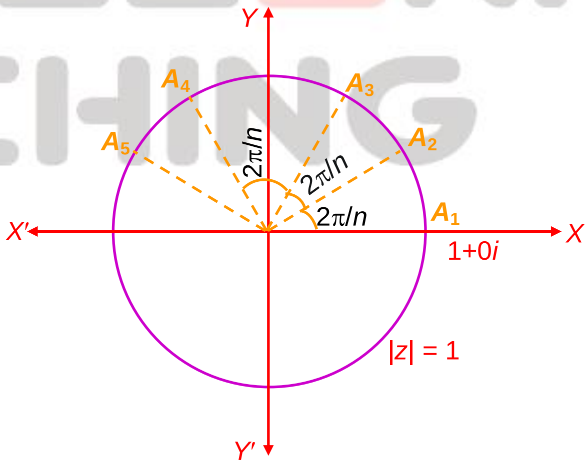
i.e. 1, α, α^2 …. α^{n-1} satisfying

(i) $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$

(ii) $\alpha^n = 1$

(iii) 1, α, ….. , α^{n-1} represent n points in the Argand plane situated on the unit circle $|Z| = 1$ and forming the corners of a regular n sides polygon.

As sum of nth roots of unity = 0



$$\Rightarrow \quad \sum_{k=0}^{n-1} \alpha^k = 0$$

$$\Rightarrow \quad \sum_{k=0}^{n-1} \left(\cos\left(\frac{2k\pi}{n}\right) + i\left(\sin \frac{2k\pi}{n}\right)\right)$$

$$\Rightarrow \quad \sum_{k=0}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = 0 \quad \text{and} \quad \sum_{k=0}^{n-1} \sin\left(\frac{2k\pi}{n}\right) = 0$$

More general equation like $x^n = a + ib$ can be solved by using this method.

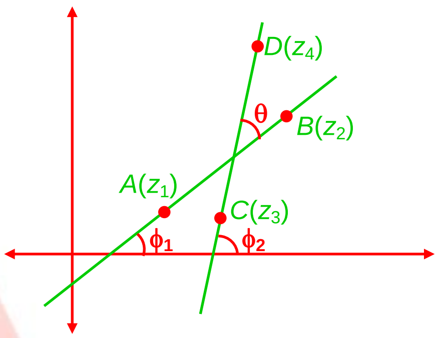
First write $a + ib = r\,[\cos\theta + i\sin\theta] = r\,[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]$ and hence the n n[th] roots of $x^n = a + ib$ are $\sqrt[n]{r}\left(\cos\dfrac{\theta + 2k\pi}{n} + i\sin\dfrac{\theta + 2k\pi}{n}\right)$; $k = 0, 1, 2, \ldots, (n-1)$

9 ROTATION THEOREM

9.1 CONI METHOD

This method gives the angle between two intersecting lines.

Let z_1, z_2, z_3 and z_4 be complex numbers representing points A, B, C, D respectively.



Then $\vec{AB} = z_2 - z_1$

$\vec{CD} = z_4 - z_3$

Let arg $\vec{AB} = \phi_1$ and arg $\vec{CD} = \phi_2$ then angle of intersection

$$\theta = \phi_2 - \phi_1 = \arg \vec{CD} - \arg \vec{AB} = \arg(z_4 - z_3) - \arg(z_2 - z_1) = \arg\left(\frac{z_4 - z_3}{z_2 - z_1}\right)$$

(i) If $\theta = 0$ or $\pm\pi$, then $\left(\dfrac{z_4 - z_3}{z_2 - z_1}\right)$ is real. Points are collinear as the two lines coincide.

It follows that if $\left(\dfrac{z_4 - z_3}{z_2 - z_1}\right)$ is real, points are collinear.

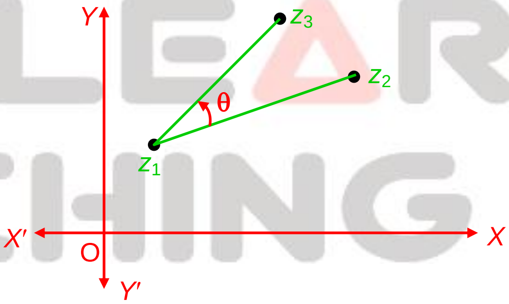
(ii) If $\theta = \pm\dfrac{\pi}{2}$, then $\left(\dfrac{z_4 - z_3}{z_2 - z_1}\right)$ is purely imaginary. It follows that if $\left(\dfrac{z_4 - z_3}{z_2 - z_1}\right)$ is purely imaginary then line joining z_1, z_2 is perpendicular to the line joining z_3, z_4.

(iii) Hence angle between the lines passing through z_2 and z_3 and intersecting at z_1 is given by

$$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \theta$$

also $z = |z|\, e^{i\theta}$

$$\Rightarrow \quad \frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| e^{i\theta}$$



9.2 CONDITION FOR FOUR POINTS TO BE CONCYCLIC

Four points z_1, z_2, z_3 and z_4 in the Argand plane are concyclic if and only if

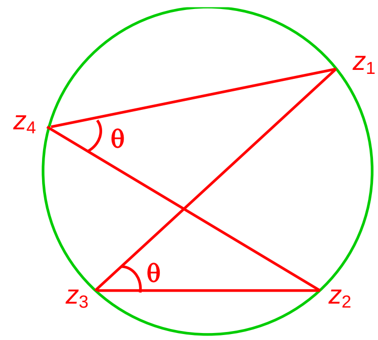
$$\arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \arg\left(\frac{z_1 - z_4}{z_2 - z_4}\right) = \theta \text{ (say)}$$

Applying coni method, we get

$$\frac{z_1 - z_3}{z_2 - z_3} = \left|\frac{z_1 - z_3}{z_2 - z_3}\right| e^{i\theta}$$

$$\frac{z_1 - z_4}{z_2 - z_4} = \left|\frac{z_1 - z_4}{z_2 - z_4}\right| e^{i\theta}$$

Solving the above two equations to eliminate θ we get



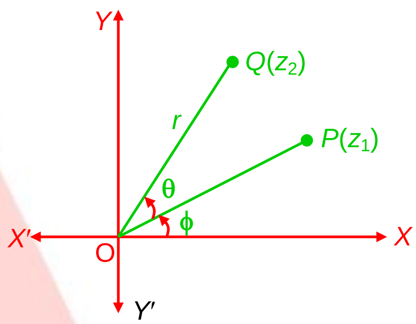
$$\frac{z_1 - z_3}{z_2 - z_3} \cdot \frac{z_2 - z_4}{z_1 - z_4} = \left|\frac{z_1 - z_3}{z_2 - z_3}\ \frac{z_2 - z_4}{z_1 - z_4}\right|$$

This is possible only if the expression on left hand side is real (it may be positive or negative, depending upon whether the points are considered in cyclic order or not).

9.3 COMPLEX NUMBER AS A ROTATING ARROW IN THE ARGAND PLANE

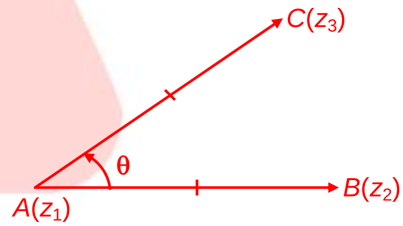
(i) If a complex number z_1 is rotated in anticlockwise sense by an angle θ and let z_2 be its new position, then $z_1 = re^{i\phi}$ and $z_2 = re^{i(\theta+\phi)}$ (as $|z_1| = |z_2| = r$)

$\Rightarrow$ $\mathbf{z_2 = z_1 e^{i\theta}}$



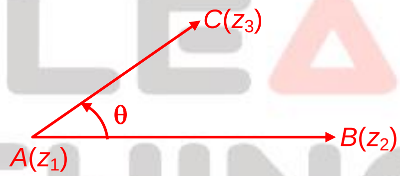
Clearly multiplication of z with $e^{i\theta}$ rotates vector $\overrightarrow{OP}$ through an angle θ in anticlockwise sense. Similarly multiplication of z with $e^{-i\theta}$ will rotate vector $\overrightarrow{OP}$ in clockwise sense.

(ii) If z_1, z_2, z_3 are the affixes of three points A, B, C such that $AC = AB$ and $\angle CAB = \theta$. Then $\overrightarrow{AC} = z_3 - z_1$ will be obtained by rotating $\overrightarrow{AB} = z_2 - z_1$ through an angle θ in anticlockwise sense and therefore



$$\mathbf{(z_3 - z_1) = (z_2 - z_1)e^{i\theta}}$$

(iii) In the above case if $AB \neq AC$, then we consider the rotation of unit vectors as



$$\frac{\mathbf{z_3 - z_1}}{\mathbf{|z_3 - z_1|}} = \frac{\mathbf{z_2 - z_1}}{\mathbf{|z_2 - z_1|}} \mathbf{e^{i\theta}}.$$

This concept has also been explained in terms of coni method earlier.

10 THEORY OF EQUATIONS WITH COMPLEX COEFFICIENTS

An n^{th} degree equation with complex coefficients a_n , a_{n-1}, ……, a_0 is given as

$$a_n z^n + a_{n-1} z^{n-1} + \ldots\ldots + a_1 z + a_0 = 0$$

It has n roots say α_1 , α_2 , ……..α_n, and

$$\sum \alpha_1 = -\frac{a_{n-1}}{a_n}$$

$$\sum \alpha_1\alpha_2 = +\frac{a_{n-2}}{a_n}$$

$$\alpha_1\alpha_2 ....\alpha_n = (-1)^n \frac{a_0}{a_n}$$

In case of quadratic equations with complex coefficients having non-zero imaginary part, discriminant has no role for existence of roots.

11 LOGARITHM OF A COMPLEX NUMBER

Let $\log_e (x + iy) = \alpha + i\beta$ …(i)

suppose $x + iy = r(\cos\theta + i \sin\theta)$

$= re^{i\theta}$ …(ii)

then $x = r\cos\theta,\ y = r\sin\theta$

so that $r = \sqrt{(x^2 + y^2)}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

from (ii) we have $\log_e (x + iy) = \log_e (re^{i\theta}) = \log_e r + \log_e e^{i\theta}$

$= \log_e r + i\theta$

$= \log_e \sqrt{(x^2 + y^2)} + i\tan^{-1}\left(\frac{y}{x}\right)$

or $\log_e(z) = \log_e |z| + i \text{ amp } z$

so, the general value is $\log(z) = \log_e(z) + 2n\pi i$ $(-\pi < \text{amp}(z) < \pi)$.

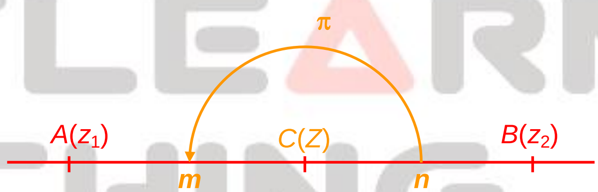
12 SECTION FORMULA

Let z_1 and z_2 represents any two complex number representing the points A and B respectively in the Argand plane. Let C be the point dividing line AB internally in ratio $m : n$ i.e. $\frac{AC}{BC} = m : n$ and let the complex number associated with point C be z.

Then let us rotate the line BC about C so that it becomes parallel to CA. Then corresponding equation after rotation will be

$$\frac{z_1 - z}{z_2 - z} = \frac{|z_1 - z|}{|z_2 - z|} e^{i\pi} = \frac{m}{n}(-1)$$

$$\Rightarrow \quad z = \frac{nz_1 + mz_2}{m + n}$$



Thus

1. If Z_1, Z_2 are divided at P in the ratio $m : n$, then P has the complex number $\frac{mZ_2 + nZ_1}{m + n}$.

   Particularly the mid point of the join of Z_1 and Z_2 is $\frac{Z_1 + Z_2}{2}$

2. If Z_1, Z_2, Z_3 be three points A, B, C forming a triangle ABC; then the centroid G of triangle ABC has an associated complex number $\frac{Z_1 + Z_2 + Z_3}{3}$.

13 LOCUS IN AN ARGAND PLANE

It has been pointed that there is a bijective correspondence between a complex number

$Z \equiv (x, y)$ and a point $P(x, y)$ in the complex plane (or) ***Argand's diagram***.

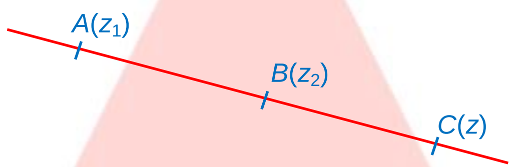
Coordinate Geometry theory gives us the concept of a locus as a curve, every point $P(x, y)$ on which satisfies a relation between x and y termed as the equation to a curve.

But $P(x, y)$ is also equivalent to $Z = (x, y)$; and hence this relation between x and y–representing the equation – can also be put in the form of a condition on Z.

To cite an example, $x^2 + y^2 = 1$, expressed in terms of Z, is $|Z| = 1$; and it is said that the condition $|Z| = 1$, being satisfied by all points Z at units distance from (0, 0), represents a circle with centre at (0, 0) and radius = 1. We therefore assert that any condition imposed on Z, automatically places a restriction on the possible locations in the Argand's diagram of the point P representing Z; and hence all such points lie on a curve. Such a curve traced in the Argand's diagram by $P \equiv Z$, because of a condition imposed on Z, is termed as **Locus in an Argand's diagram**

13.1 STRAIGHT LINE

Equation of straight line passing through points. A and B represented by complex numbers z_1 and z_2 is



Let us take $C(z)$ is the general point on line then

$$\arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0 \quad \text{or} \quad \pi$$

$$\Rightarrow \qquad \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$

It can also be represented in the following form

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

The general form of straight line is

$$a\bar{z} + \bar{a}z + b = 0$$

where a is a complex number and b is real.

Slope of a line

Let equation of line be $z\bar{a} + \bar{z}a + b = 0$

Replacing z by x + iy, we get,

$$(a + \bar{a})x + i(\bar{a} - a)y + b = 0$$

Its real slope is = $\dfrac{a + \bar{a}}{i(\bar{a} - a)} = \dfrac{-\text{Re}(a)}{\text{Im}(a)}$

Its complex slope is $-\dfrac{a}{\bar{a}} = -\dfrac{coeff\ \ of\ \ \bar{z}}{coeff\ \ of\ \ z}$

Equation of the line parallel to $\bar{a}z + a\bar{z} + b$ is $\bar{a}z + a\bar{z} + \lambda = 0$ (where λ is a real number) and that of line perpendicular to it is $z\bar{a} - \bar{z}a + i\lambda = 0$.

Ray

- arg $Z = \theta$ is a ray (or a straight line) from the origin and pointed in such a direction that any point Z situated on the lie has an argument θ
- arg $(z - \alpha) = \theta$ is a ray (or a straight line) from the point α and pointed in such a direction that the join of α to Z is inclined at an angle θ to the positive direction of the real axis (x-axis)

Perpendicular bisector

- $|Z-\alpha| = |Z-\beta|$ represents the perpendicular bisector of the join of the two points $\alpha \equiv (p, q)$ and $\beta \equiv (r, s)$.
- Perpendicular distance of a point z_0 from line $\bar{a}z + a\bar{z} + b = 0$ is $= \dfrac{|\bar{a}z_0 + a\bar{z}_0 + b|}{2|a|}$.

13.2 CIRCLE

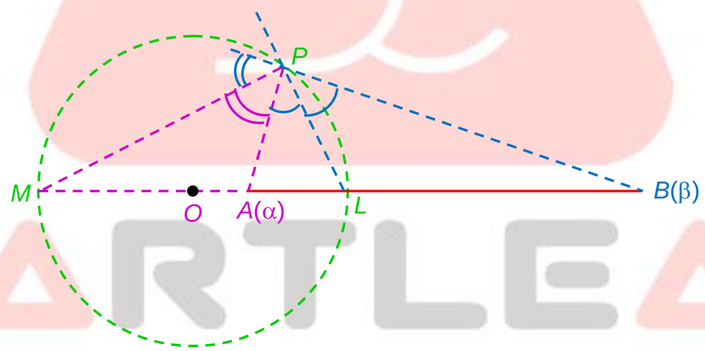
- $|Z| = r$ is a circle, centre (0, 0) and radius r.
- $|Z-\alpha| = r$ (α, complex) is a circle, centre at $\alpha \equiv (p, q)$ and radius $= r$ since $|Z-\alpha|$ represents the absolute distance of Z from α.
- $|Z-\alpha| = k|Z-\beta|$ (k real > 0, $\neq 1$) is the circle any point P on which, with reference to the points $A(\alpha)$ and $B(\beta)$, satisfies the condition $\dfrac{AP}{PB} = k\ (k \neq 1)$

  Let us take, for exactness, $0 < k < 1$. Let L and M divide the join of $A(\alpha)$; $B(\beta)$ internally at L and externally at M in the ratio k, so that $\dfrac{AL}{LB} = \dfrac{MA}{MB} = k < 1$

  Draw the circle with LM as diameter. Any point P on this circle will satisfy the requirement $\dfrac{AP}{PB} = k$.

  The locus of the point $P(Z)$ satisfying the condition $\dfrac{|Z-\alpha|}{|Z-\beta|} = k \neq 1$ is this circle on

  LM as diameter; and is called the **Appolonius Circle** of A and B with respect to the ratio k. The circle meets AB at L and M and these two points, being points on the circle, trivially satisfy the condition to be satisfied by any point P to lie on the circle. In fact the choice of L and M have been made to satisfy this requirement. It may be also pointed, as a property, that PL and PM bisect $\angle APB$ internally and externally.



- $\arg\left(\dfrac{Z-Z_1}{Z-Z_2}\right) = 0$ is a straight line – that part of the segment of the line through Z_1 and Z_2 which is outside the segment joining Z_1 and Z_2
- $\arg\left(\dfrac{Z-Z_1}{Z-Z_2}\right) = \pi$ represent the line segment joining Z_1 and Z_2.

  In fact the condition $\arg\left(\dfrac{Z_1-Z_2}{Z_1-Z_3}\right) = 0$ or π is the condition for Z_1, Z_2 and Z_3 to be collinear.

- $\arg\left(\dfrac{Z-Z_1}{Z-Z_2}\right) = \theta \neq 0 \neq \pi$. The equation $\arg\left(\dfrac{Z-Z_1}{Z-Z_2}\right) = \theta$ geometrically expresses the fact that the join of Z_1 and Z_1 subtends at Z, the angle θ. Hence the condition represents the segment of a circle described on the join if Z_1 and Z_2 as a chord and containing at any point $P(Z)$ on the segment the angle θ. If $0 < \theta < \pi/2$, the segment is a major segment. If $\pi/2 < \theta < \pi$, the segment is a minor segment. If $\theta = \pi/2$ the locus is the semi-circle on the join of Z_1 and Z_1 the circle being appropriately chosen.

It has already been pointed out that every point can be taken to be represented by a complex number Z. Thus just as in Coordinate Geometry where we have for every point a pair of numbers (its coordinates), in complex number theory every point has an associate complex number, of which, the point is but a geometrical representation.

13.3 CONIC SECTION

Parabola

Equation of parabola with focus at z_0 and directrix as $\bar{a}z + a\bar{z} + b = 0$ is given by

$$|z - z_0| = \frac{|a\bar{z} + \bar{a}z + b|}{2|a|}$$

Ellipse

Equation of ellipse with foci at z_1 and z_2 and length of major axes as $2a$ is

$$|z - z_1| + |z - z_2| = 2a$$

where $2a > |z_1 - z_2|$

Hyperbola

Equation of hyperbola with foci at z_1 and z_2 and length of transverse axes as $2a$ is

$$\big||z - z_1| - |z - z_2|\big| = 2a$$

where $2a < |z_1 - z_2|$