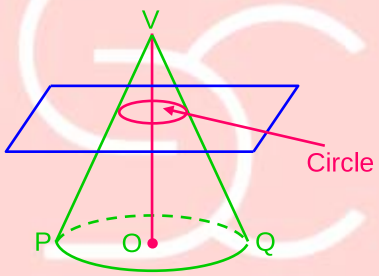
Conic Sections

1 CONIC SECTION

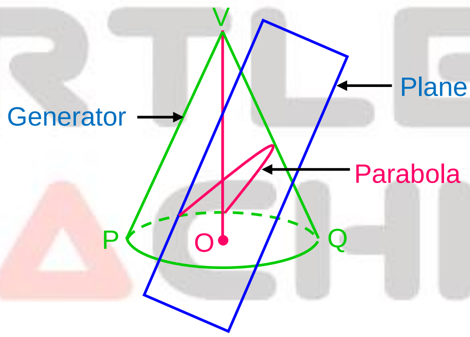
Take two sticks tied at one of the ends. Keep one stick fixed and rotate the other around it by 360°. The figure so generated is a cone. The fixed stick is the axis and the one moved around is referred to as the generator of the cone.

And when a plane cuts the cone at different angles we get different sections of the cone namely circle, parabola, ellipse and hyperbola.

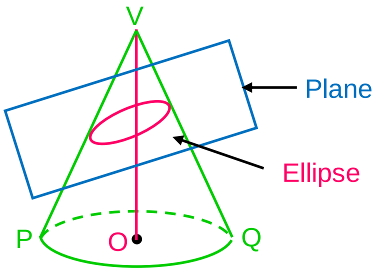
Now if a plane cuts the cone such that it is perpendicular to the axis of the cone, the section is a circle.



Section of a right circular cone cut by a plane, which is parallel to a generator of the cone is a parabola.

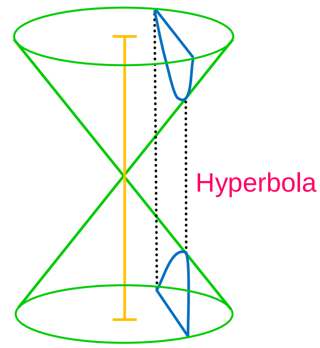


Section of a right circular cone by a plane which is not parallel to any generator and not parallel or perpendicular to the axis of the cone is an ellipse.



Section of a right circular cone by a plane which is parallel to the axis of the cone is a hyperbola.

Geometrically, a parabola, ellipse and hyperbola are expressed in terms of a fixed point called focus and a fixed line called directrix.

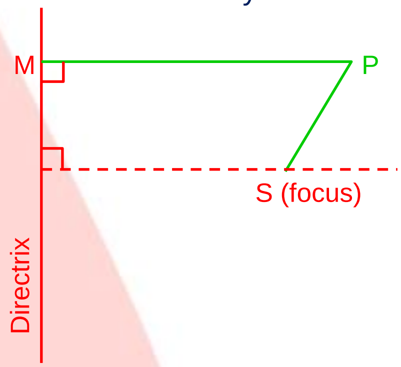


A conic section or a conic is the locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is always a constant quantity.

The fixed point is called the **focus** of the conic and the fixed line is called the **directrix** of the conic. Also this constant ratio is called the **eccentricity** of the conic and is denoted by ***e***.

If $e = 1$, the conic is called parabola.
If $e < 1$, the conic is called ellipse.
If $e > 1$, the conic is called hyperbola.
If $e = 0$, the conic is called circle.
If $e \to \infty$, the conic is called a pair of straight lines.
In the figure

$$\frac{SP}{PM} = \text{constant} = e$$

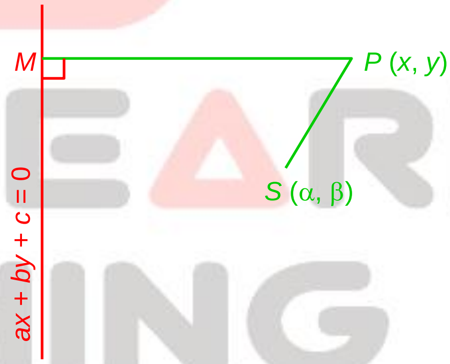


or $SP = e\,(PM)$

If the focus is (α, β) and the directrix is $ax + by + c = 0$, then the equation of the conic section whose eccentricity e given as

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = e.\frac{|ax+by+c|}{\sqrt{(a^2+b^2)}}$$

or $$(x-\alpha)^2 + (y-\beta)^2 = e^2.\frac{(ax+by+c)^2}{(a^2+b^2)}$$



Axis: The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.
Vertex: The points of intersection of the conic section and the axis is (are) called vertex (vertices) of the conic section.
Focal chord: Any chord passing through the focus is called focal chord of the conic section.
Double ordinate: A straight line drawn perpendicular to the axis and terminated at both ends of the curve is a double ordinate of the conic section.
Latus rectum: The double ordinate passing through the focus is called the latus rectum of the conic section.
Centre: The point which bisects every chord of the conic passing through it, is called the centre of the conic section.

The general equation of second degree in x, y is

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ ...(i)

it will represent *a* conic if

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

further (i) will represent

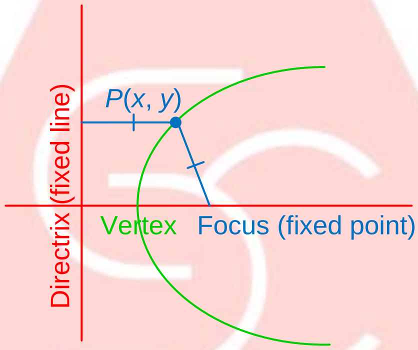
an ellipse if $h^2 < ab$

a parabola if $h^2 = ab$

a hyperbola if $h^2 > ab$

2 PARABOLA

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point is equal to its perpendicular distance from a fixed line. The fixed point is called focus and the fixed line is called the directrix. The line through focus perpendicular to directrix is called the axis. Mid-point of point of intersection of axis and directrix and focus is called the vertex.

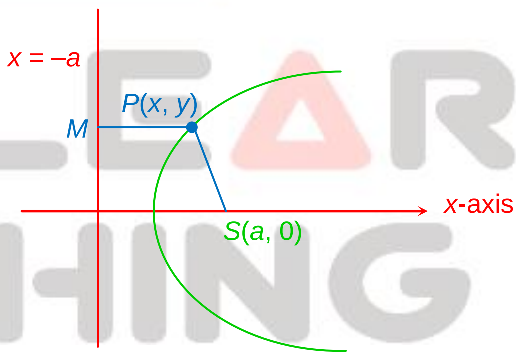


3 STANDARD EQUATION OF A PARABOLA

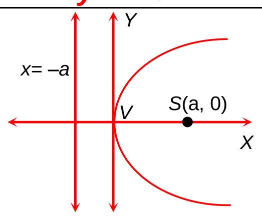
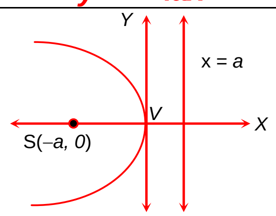
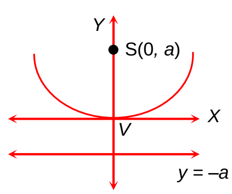
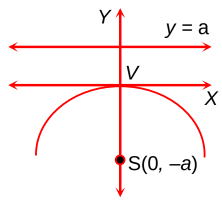
If we take axis of the parabola as x-axis with vertex as origin, focus as $S(a, 0)$, then clearly directrix will be $x + a = 0$.
Now if $P(x, y)$ is a general point on the parabola, then by definition $PS = PM$.

$\Rightarrow \quad (x - a)^2 + y^2 = (x + a)^2$

$\Rightarrow \quad y^2 = 4ax$



There are other three forms also. All the four forms are listed as follows:

Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Graph				
Vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)

Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Equation of tangent at the vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$

If the vertex is not the origin and the axis and directrix are parallel to the coordinate axes, then the equation of parabola with vertex at (h, k) can be obtained by using translation of axes as follows:

Form	Vertex	Focus	Equation of directrix	Equation of axis	Tangent at the vertex
$(y-k)^2 = 4a(x-h)$	(h, k)	$(h + a, k)$	$x = h - a$	$y = k$	$x = h$
$(y-k)^2 = -4a(x-h)$	(h, k)	$(h - a, k)$	$x = h + a$	$y = k$	$x = h$
$(x-h)^2 = 4a(y-k)$	(h, k)	$(h, k + a)$	$y = k - a$	$x = h$	$y = k$
$(x-h)^2 = -4a(y-k)$	(h, k)	$(h, k - a)$	$y = k + a$	$x = h$	$y = k$

Note : in all the above cases $a > 0$.

4 PARAMETRIC FORM OF A PARABOLA

Let $y^2 = 4ax$ be a parabola, then for any real t, $x = at^2$, $y = 2at$ satisfy the equation of parabola, so point $(at^2, 2at)$ lie on the parabola. Here $x = at^2$, $y = 2at$ is known as parametric form of the parabola. In short, we denote the point $P(at^2, 2at)$ as $P(t)$.

5 IMPORTANT CONCEPTS IN PARABOLA

5.1 FOCAL DISTANCE

Distance of any point P on the parabola from its focus is known as focal distance of the point P. The focal distance of any point on the parabola is equal to sum of one fourth of length of latus rectum and its distance from the tangent at the vertex. In simple words, focal distance of $P(x, y)$, a point on the parabola $y^2 = 4ax$, is equal to $|x + a|$ as

$$PS = \sqrt{(x-a)^2 + (y-0)^2} = \sqrt{(x-a)^2 + 4ax} = \sqrt{(x+a)^2} = |x + a|$$

5.2 FOCAL CHORD

Any chord of the parabola passing through its focus is known as focal chord. If one end of focal chord is $(at^2, 2at)$, then other end is given by $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

5.3 DOUBLE ORDINATE

Any chord of the parabola $y^2 = 4ax$, which is parallel to its directrix or in other words perpendicular to axis of parabola is known as double ordinate.

5.4 LATUS RECTUM

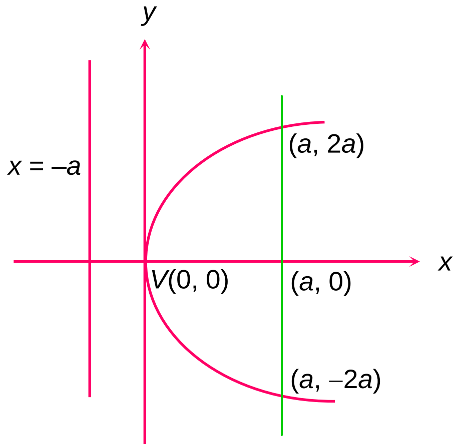
A focal chord of the parabola, which is parallel to its directrix is known as latus rectum. In parabola $y^2 = 4ax$, end point of latus rectum are $(a, 2a)$ and $(a, -2a)$ and length of latus rectum is $4a$. Two parabola are said to be equal if the length of their latus rectums are equal.

To obtain end points of a latus rectum, putting $x = a$ in

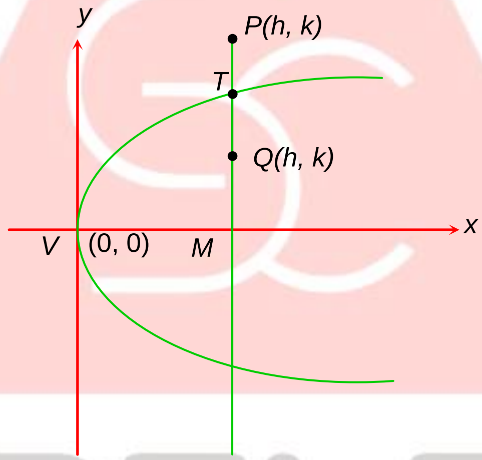
$y^2 = 4ax$

$y^2 = 4a^2$

$\Rightarrow \quad y = \pm 2a$



6 POSITION OF A POINT WITH RESPECT TO THE PARABOLA



Let $y^2 = 4ax$ be a parabola and $P(h, k)$ be a general point.

If $k^2 = 4ah$ *i.e.*, $k^2 - 4ah = 0$, then point lie on the parabola.

If $P(h, k)$ lie out sides the parabola.

$PM > TM$

$\Rightarrow PM^2 > TM^2$

$\Rightarrow k^2 > 4ah$

If $Q(h, k)$ lie inside the parabola, then $QM < TM$

$\Rightarrow QM^2 < TM^2$

$\Rightarrow k^2 < 4ah$

so, if $k^2 - 4ah < 0$, point lie inside the parabola

$k^2 - 4ah = 0$, point lie on the parabola

$k^2 - 4ah > 0$, point lie outside the parabola.

7 INTERSECTION OF A STRAIGHT LINE WITH THE PARABOLA

Consider the parabola $y^2 = 4ax$ and the straight line $y = mx + c$, if we eliminate y from both the equation, we get

$$m^2x^2 + 2(mc - 2a)x + c^2 = 0 \qquad \text{...(i)}$$

the roots of (i) will give the abscissa of the points, where line and the parabola meet.

Discriminant of (i) will be

$D = 4(mc-2a)^2 - 4m^2c^2 = 16\,(a^2 - amc)$

Now we have following results:

(i) If $a < mc$, then line $y = mx + c$ is an imaginary chord of the parabola $y^2 = 4ax$.

(ii) If $a = mc$, line is a tangent to the parabola.

(iii) If $a > mc$, line is a chord of the parabola.

(iv) In result (iii), the length of the chord is given by $\frac{4}{m^2}\sqrt{1+m^2}\sqrt{a(a-mc)}$.

(v) If m is very-very small *i.e.,* line is almost parallel to axis of the parabola. One root of equation (i) is very-very large. Hence, in this case line cuts the parabola at one point at infinity so cuts the parabola only at one point.

8 TANGENT TO A PARABOLA

8.1 TANGENT AT THE POINT (x_1, y_1)

Consider the parabola $y^2 = 4ax$, the equation of tangent at $P(x_1, y_1)$ on the parabola is given by $T = 0$

$$i.e.,\ yy_1 - 4a\left(\frac{x + x_1}{2}\right) = 0$$

or $yy_1 - 2ax - 2ax_1 = 0$

8.2 TANGENT AT THE POINT WITH PARAMETER '*t*'

Let $P(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$. Then equation of tangent at P is given by

$y \,.\, 2at = 2a(x + at^2)$

or $ty = x + at^2$

8.3 TANGENT IN THE SLOPE FORM

In 8.2, slope of tangent at point 't' is $\frac{1}{t}$. If we put $\frac{1}{t} = m$.

We get equation of tangent in the slope form as $y = mx + \frac{a}{m}$ and here point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$. We do not have any tangent to parabola, which is parallel to its axis.

8.4 POINT OF INTERSECTION OF TANGENTS

Point of intersection of tangents at the points 't_1' and 't_2' is $(at_1t_2, a(t_1 + t_2))$.

8.5 EQUATION OF THE TANGENTS FROM AN EXTERNAL POINT

Consider the parabola $y^2 = 4ax$ and let $P(h, k)$ be an external point, then combined equation of tangents from P to parabola is given $T^2 = SS_1$ *i.e.*,

$(y^2 - 4ax)(k^2 - 4ah) = (yk - 2a(x + h))^2$

.

8.6 CHORD OF CONTACT

From an external point P (h, k), let PT_1 and PT_2 be two tangents drawn to parabola $y^2 = 4ax$, then line joining point of contacts T_1 and T_2 is known as chord of contact, its equation is given by $T = 0$.

i.e., $ky - 2a(x + h) = 0$.

9 CHORD OF THE PARABOLA WITH MID POINT

Consider the parabola $y^2 = 4ax$ and let $P(h, k)$ be an internal point of the parabola, then equation of chord with $P(h, k)$ as mid-point is given by $T = S_1$, *i.e.,*

$ky - 2a(x + h) = k^2 - 4ah.$

10 NORMAL TO THE PARABOLA

10.1 NORMAL AT THE POINT (x_1, y_1)

Slope of tangent at (x_1, y_1) to the parabola $y^2 = 4ax$ is $\frac{2a}{y_1}$, so slope of normal will be $-\frac{y_1}{2a}$.

Hence, equation of normal will be $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

10.2 NORMAL AT THE POINT '*t*'

Let $P(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$, then equation of normal at P is given by

$$y - 2at = -\frac{2at}{2a}(x - at^2)$$

or $y + tx = 2at + at^3$

If this normal passes through the point (h, k), then

$k + th = 2at + at^3$

or $at^3 + (2a - h)t - k = 0$ ...(i)

if t_1, t_2, t_3 be the roots of the equation (i), then

$t_1 + t_2 + t_3 = 0$ ...(ii)

$t_1 t_2 + t_1 t_3 + t_2 t_3 = \frac{2a - h}{a}$ ...(iii)

$t_1 t_2 t_3 = \frac{k}{a}$ ...(iv)

So, in general three normals can be drawn from a given point (h, k) to the parabola.

If we multiply (ii) by $2a$, (iii) by $4a^2$ and (iii) by $8a^3$, we get

$y_1 + y_2 + y_3 = 0$

$y_1y_2 + y_1y_3 + y_2y_3 = 4a(2a - h)$

$y_1y_2y_3 = 8a^2k$

Where y_1, y_2, y_3 are the ordinates of the feets of normals drawn from the point (h, k) to the parabola $y^2 = 4ax$. Then we have the following results:

If normals at 3 points on parabola $y^2 = 4ax$ are concurrent, then sum of the ordinates of the feets of normals is zero.

10.3 NORMAL IN TERMS OF SLOPE

If in 10.2, we put the slope of normal $-t = m$, we get equation of normal to parabola $y^2 = 4ax$ at point $(am^2, -2am)$ as $y - mx + 2am + am^3 = 0$.

If this normal passes through (h, k), then

$am^3 + (2a - h)m + k = 0$ ...(i)

roots of (i) will give slopes of normal which are drawn from the point (h, k)

$m_1 + m_2 + m_3 = 0$

$m_1 m_2 + m_1 m_3 + m_2 m_3 = \frac{2a - h}{a}$

$m_1 m_2 m_3 = \frac{-k}{a}$

Here we note that if three normals to parabola $y^2 = 4ax$ are concurrent, then sum of the slopes of normals is zero.

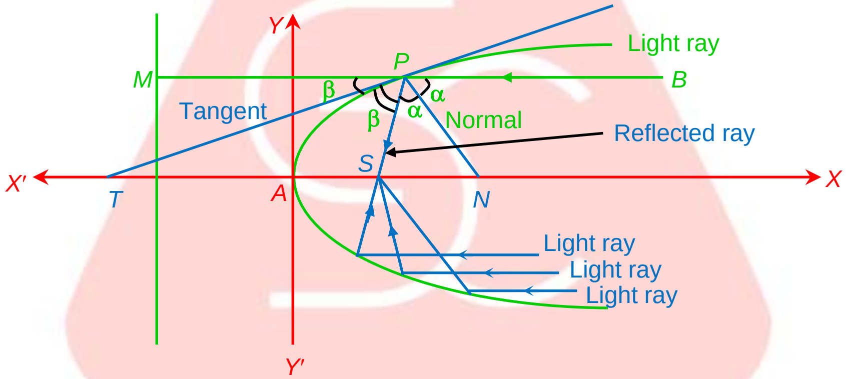
10.4 POINT OF INTERSECTION OF NORMALS

If normal at the point 't_1' to the parabola $y^2 = 4ax$, meets again to parabola at the point 't_2', then $t_2 = -t_1 - \frac{2}{t_1}$. If normals at the point 't_1' and 't_2' intersect on parabola at point 't_3' itself, then $t_1t_2 = 2$ and $t_3 = -t_2 - t_1$. The point of intersection of normals at the point 't_1' and 't_2' is $(2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2))$.

11 REFLECTION PROPERTY OF THE PARABOLA

Reflection property of a parabola

The tangent (PT) and normal (PN) of the parabola $y^2 = 4ax$ at P are the internal and external bisectors of ∠SPM and BP is parallel to the axis of the parabola and ∠BPN = ∠SPN.



ELLIPSE

1 DEFINITION AND EQUATION

1.1 DEFINITION 1

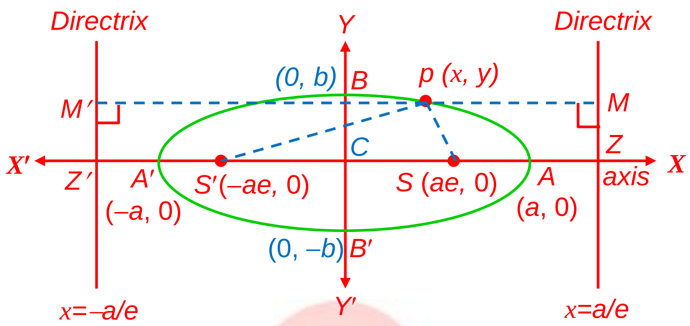
An ellipse is the locus of a point, which moves such that the ratio of its distances from a fixed point to a fixed line is constant and value of the constant is less than unity. Here, fixed point is called as focus, fixed line as directrix and constant as eccentricity.

Generally, the equation of the an ellipse, whose focus is the point (h, k) and directrix $lx + my + n = 0$ and whose eccentricity is e, is

$$(x-h)^2 + (y-k)^2 = e^2 . \frac{(lx+my+n)^2}{(l^2+m^2)} \quad (e < 1)$$

1.2 STANDARD EQUATION

If we take focus as $(\pm ae, 0)$ and directrix as $x = \pm\frac{a}{e}$, we get equation of ellipse as



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where $b^2 = a^2(1 - e^2)$

Results

(i) Equation of the directrix corresponding to focus $(-ae, 0)$ is $x=\frac{-a}{e}$

(ii) If $a > b$, $2a$ is the length of major axis and $2b$ is the length of minor axis.

Centre

Point of intersection of major axis and minor axis is the centre of the ellipse. In the above case centre ≡ (0, 0).

Focal chord

Any chord of the ellipse passing through the focus is called focal chord.

Latus rectum

A focal chord, which is perpendicular to the major axis of the ellipse is known as latus-rectum.

In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b)$,

(i) End points of the latus rectum are $\left(\pm ae, \pm\frac{b^2}{a}\right)$.

(ii) Length of the latus rectum is $\frac{2b^2}{a}$.

Focal distance of a point

Let P be a point on the ellipse, where S and S′ are the foci, then PS and PS′ are known as focal distance of the point P.

$$PS = ePM = e\left(\frac{a}{e}-x\right)=a-ex \quad \text{and} \quad PS' = e\,PM'$$

$$PS' = e\left(\frac{a}{e}+x\right)=a+ex$$

Remark:

In any ellipse sum of the focal distances of any point is constant and is equal to length of major axis.

1.3 DEFINITION 2

An ellipse is the locus of a point, which moves such that sum of its distances from two fixed points is constant and value of the constant is more than distance between these two fixed points. These two fixed points are two foci of the ellipse and sum of distances is equal to length of its major axis.

Remark:

For the ellipse $\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1 \quad (a > b)$, Eccentricity = $e = \sqrt{1 - \frac{b^2}{a^2}}$

Equation of major axis ≡ $y = \beta$
Equation of minor axis ≡ $x = \alpha$
Centre ≡ (α, β)
Vertices ≡ $(\alpha \pm a, \beta)$
Foci ≡ $(\alpha \pm ae, \beta)$

Directrices ≡ $x = \alpha \pm \frac{a}{e}$

Ends of minor axis ≡ $(\alpha, \beta \pm b)$

Length of latus rectum = $\frac{2b^2}{a}$

Parametric coordinates ≡ $(a\cos\theta + \alpha, b\sin\theta + \beta)$ (where $0 \le \theta < 2\pi$).

$$4(x^2 + 4y^2 + 1 - 4xy + 2x - 4y) + 9(4x^2 + y^2 + 4 + 4xy + 8x + 4y) = 25$$

or, $40x^2 + 20xy + 25y^2 + 80x + 20y + 15 = 0$

or, $8x^2 + 4xy + 5y^2 + 16x + 4y + 3 = 0$.

Comparing with the general equation of the second degree

$h^2 - ab = 2^2 - 8.5 = 4 - 40 < 0$

$\Delta = 8.5.3 + 2.2.8.2 - 8.2^2 - 5.8^2 - 3.2^2$

$= 120 + 64 - 32 - 320 - 12 \neq 0$, $h \neq 0$

∴ the curve is an ellipse.

For the equation $4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 25$, we find that $x - 2y + 1 = 0$ and $2x + y + 2 = 0$ are mutually perpendicular lines.

∴ substituting $\frac{x - 2y + 1}{\sqrt{1^2 + (-2)^2}} = X$ …(i)

and $\frac{2x + y + 2}{\sqrt{2^2 + 1^2}} = Y$ …(ii)

the equation changes to $4(\sqrt{5}X)^2 + 9(\sqrt{5}Y)^2 = 25$

or, $4X^2 + 9Y^2 = 5$

or $\frac{X^2}{5/4} + \frac{Y^2}{5/9} = 1$

which is in the standard form of equation of an ellipse.

∴ $a^2 = \frac{5}{4}$ and $b^2 = \frac{5}{9}$;

but $b^2 = a^2(1 - e^2)$

∴ $\frac{5}{9} = \frac{5}{4}(1 - e^2)$

or $\frac{4}{9} = 1 - e^2$

∴ $e^2 = 1 - \frac{4}{9} = \frac{5}{9}$;

∴ $e = \frac{\sqrt{5}}{3}$

Now, centre = $(0, 0)_{X,\,Y}$;

when $X = 0$, $Y = 0$ we have from (i), (ii), $x - 2y + 1 = 0$ and $2x + y + 2 = 0$

Solving these, $5x + 5 = 0$

∴ $x = -1$; so $y = 0$

∴ centre = (–1, 0).

2 POSITION OF A POINT RELATIVE TO AN ELLIPSE

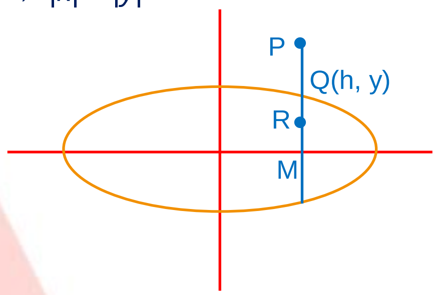
Let the equation of the ellipse be $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

then point $P(h, k)$ lies outside the ellipse, if $PM > QM$ $\Rightarrow$ $|k| > |y|$

$\Rightarrow$ $k^2 > y^2$

$\Rightarrow$ $k^2 > b^2\left(1 - \frac{h^2}{a^2}\right)$

$\Rightarrow$ $\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1 > 0$ i.e. $S_1 > 0$



Similarly R lies inside the ellipse if $RM < QM$

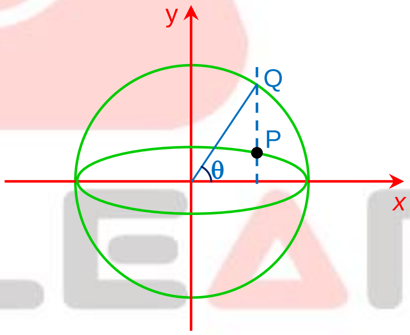
$\Rightarrow$ $\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1 < 0$ i.e. $S_1 < 0$

3 PARAMETRIC EQUATION OF ELLIPSE

If we draw a circle with major axis of any ellipse as diameter then this circle is known as auxiliary circle for that ellipse. Now take any point P on the ellipse and draw a line through it parallel to minor axis. The point where this line cuts the auxiliary circle such that P and Q lies on the same side of the major axis, is known as corresponding point. If the line joining to the center of the ellipse, makes an angle of θ with the major axis, θ is known as eccentric angle of point P.

If ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$, then auxiliary circle will be $x^2 + y^2 = a^2$ and point Q will be $(a\cos\theta, a\sin\theta)$, hence point P will be $(a\cos\theta, b\sin\theta)$.

Here $x = a\cos\theta$, $y = b\sin\theta$ is called as parametric equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



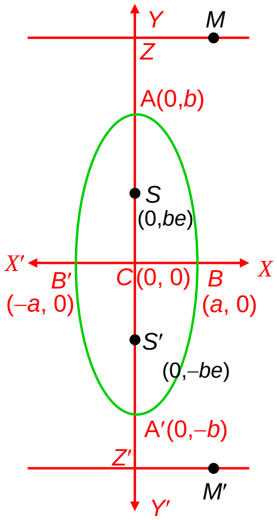
4 ANOTHER FORM OF ELLIPSE (when b > a)

In this case major and minor axis of the ellipse along y-axis and x-axis respectively.
then $AA' = 2b$ (length of major axis) and $BB' = 2a$ (length of minor axis)
The foci S and S' are $(0, be)$ and $(0, -be)$ respectively.

The directrices are MZ and $M'Z'$ are $y = \frac{b}{e}$ and $y = -\frac{b}{e}$ respectively.

$$e^2 = 1 - \frac{a^2}{b^2}$$

Length of latus rectum = $\frac{2a^2}{b}$



HYPERBOLA

1 DEFINITION AND EQUATION

1.1 DEFINITION 1

A hyperbola is the locus of a point, which moves such that ratio of its distance from a fixed point to a fixed line is constant and value of the constant is greater than unity. Here, fixed point is known as focus, fixed line as directrix and constant as eccentricity.
Generally, the equation of the hyperbola, whose focus is the point (h, k), directrix is $lx + my + n = 0$ and eccentricity e, is

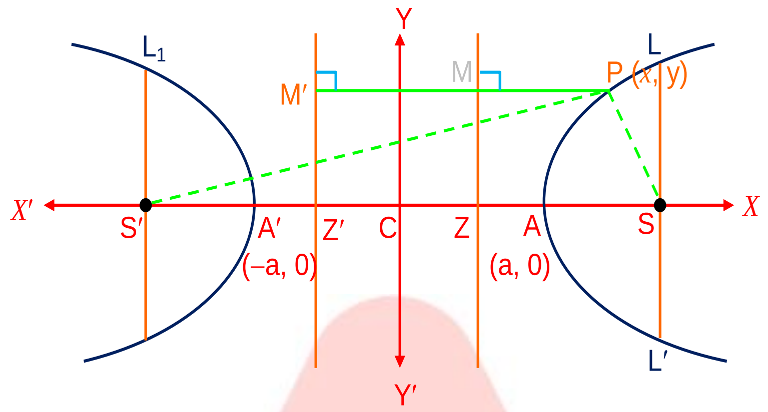
$$(x-h)^2 + (y-k)^2 = e^2 . \frac{(lx+my+n)^2}{(l^2+m^2)} \qquad (e > 1)$$

1.2 DEFINITION 2

A hyperbola is the locus of a point, which moves such that the difference in its distances from two fixed points is constant and value of the constant is less than the distance between these two fixed points.

1.3 STANDARD EQUATION

If we take focus as $(\pm ae, 0)$ and directrix as $x = \pm\frac{a}{e}$, we get equation of hyperbola as



$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

where $b^2 = a^2 (e^2 - 1)$

Results

(i) Equation of the directrix corresponding to focus $(-ae, 0)$ is $x=\frac{-a}{e}$

(ii) $2a$ is the length of transverse axis and $2b$ is the length of conjugate axis.

Focal chord

Any chord of the hyperbola passing through the focus is called focal chord.

Latus rectum

A focal chord, which is perpendicular to the transverse axis of the hyperbola is known as latus-rectum.

In a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $(a > b)$,

(i) End points of the latus rectum are $\left(\pm ae, \pm\frac{b^2}{a}\right)$

(ii) Length of the latus rectum is $\frac{2b^2}{a}$

Focal distance of a point

Let P be a point on the hyperbola, where S and S′ are the foci, then PS and PS′ are known as focal distance of the point P.

Remark:

In any hyperbola difference in the focal distances of any point is constant and is equal to length of transverse axis.

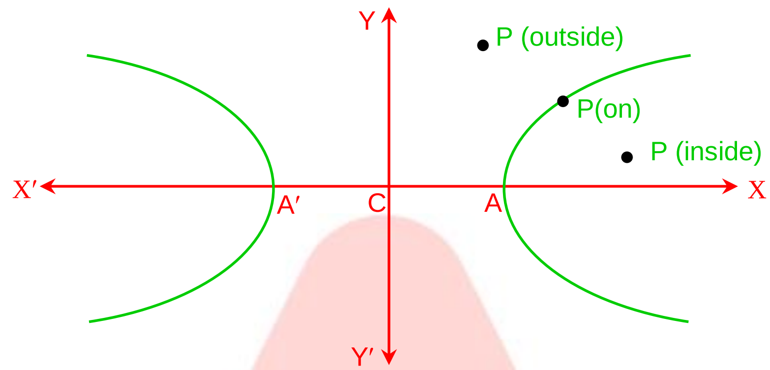
2 POSITION OF A POINT RELATIVE TO A HYPERBOLA

Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Let $P \equiv (h, k)$

Now P will lie outside, on or inside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as

$$\frac{h^2}{a^2}-\frac{k^2}{b^2}-1<,=,>0$$



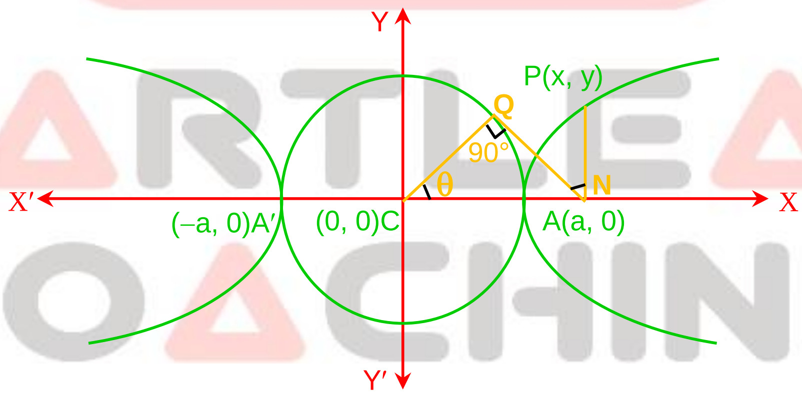
Note: Terms 'inside' and 'outside' are the conventions taken here, as these terms are defined only for closed curves only.

3 PARAMETRIC EQUATION OF HYPERBOLA

Take any point *P* on the hyperbola and draw a line through it parallel to conjugate axis. From the point where it cuts the transverse axis, draw a tangent to the auxiliary circle. Now the angle made by the line joining the point of tangency to the centre of the circle from the positive *x*-axis is known as eccentric angle of the point *P*.

If hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$,

Then parametric coordinates of point *P* will be ($a\sec\theta$, $b\tan\theta$), here 'θ' is the eccentric angle of point *P*.



Here $x = a\sec\theta$, $y = b\tan\theta$ is called parametric equation of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

4 CONJUGATE HYPERBOLA

A hyperbola is called conjugate hyperbola of a given hyperbola, which is obtained by interchanging transverse and conjugate axes of the given hyperbola.

If $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be a given hyperbola then equation of its conjugate hyperbola will be given as

$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=-1.$

5 TANGENT

5.1 POINT FORM

Let $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ be a hyperbola and (x_1, y_1) be a point on it. Then equation of tangent to the hyperbola at the point (x_1, y_1) is $\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}-1=0$ written as $T = 0$.

5.2 PARAMETRIC FORM

Let $(a\sec\theta, b\tan\theta)$ be a point on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, then equation of tangent to the hyperbola is $\dfrac{x}{a}\sec\theta-\dfrac{y}{b}\tan\theta-1=0$

5.3 SLOPE FORM

The line $y = mx + c$ will touch the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ if and only if $c^2 = a^2 m^2 - b^2$. So equation of tangent to the hyperbola with slope m is $y=mx\pm\sqrt{a^2m^2-b^2}$. Here '±' denotes that we can get two tangents to the hyperbola with the same slope m.

5.4 DIRECTOR CIRCLE

The locus of the point where perpendicular tangents of the hyperbola meet, is called director circle of the hyperbola.

Let the tangent $y=mx\pm\sqrt{a^2m^2-b^2}$ passes through the point (h, k) then,

$k=mh\pm\sqrt{a^2m^2-b^2} \quad\Rightarrow\quad (k-mh)^2 = a^2 m^2 - b^2$

$\Rightarrow \quad (h^2 - a^2)\, m^2 - 2kmh + k^2 + b^2 = 0$ …(i)

Roots of the equation will give slope of the tangents which are intersecting at (h, k). For director circle, $m_1m_2 = -1$.

$\Rightarrow \quad \dfrac{k^2+b^2}{h^2-a^2}=-1 \quad\Rightarrow\quad h^2 + k^2 = a^2 - b^2$

So locus of point (h, k) is $x^2 + y^2 = a^2 - b^2$, which is the equation of the director circle of the hyperbola.

So in general director circle of any hyperbola is a circle with the centre same as centre of the hyperbola and radius as square root of difference in the squares of length of semi transverse axis and semi conjugate axis.

5.5 EQUATION OF TANGENTS FROM AN EXTERNAL POINT (x_1, y_1)

Let (x_1, y_1) be a point outside the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ then combined equation of the tangents to the hyperbola is $\left(\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}-1\right)^2=\left(\dfrac{x_1^2}{a^2}-\dfrac{y_1^2}{b^2}-1\right)\left(\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-1\right)$ written as $T^2 = SS_1$.

5.6 CHORD OF CONTACT

Let (x_1, y_1) be a point outside the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, then the equation of chord of contact of the point with respect to the hyperbola is $\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}-1=0$, written as $T = 0$.

6 NORMAL

6.1 POINT FORM

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be a hyperbola and $P(x_1, y_1)$ be a point on it. Then the equation of normal to the hyperbola at the point P is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$.

6.2 PARAMETRIC FORM

Let P($a\sec\theta$, $b\tan\theta$) be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then equation of normal to the hyperbola at the point P is $ax\cos\theta + by\cot\theta = a^2 + b^2$

6.3 SLOPE FORM

The line $y = mx + c$ will be a normal to the hyperbola if and only if $c=\frac{\pm m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$. So equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with slope m is given by $y=mx\pm\frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$.

7 CHORDS OF HYPERBOLA

Consider a line $y = mx + c$ and the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ by eliminating y from both the equation we get

$$(a^2m^2-b^2)x^2+2a^2mcx+a^2c^2+a^2b^2=0 \qquad \ldots\text{(i)}$$

Roots of one will given abscissa's of the point where straight line cuts the hyperbola.

Discriminant of equation (i) is

$$D=4a^4m^2c^2-4a^2(c^2+b^2)(a^2m^2-b^2)$$
$$=4a^2\{a^2m^2c^2-a^2m^2c^2-b^2a^2m^2+b^4+b^2c^2\}$$
$$=4a^2b^2\{c^2+b^2-a^2m^2\}$$

(i) If $c^2 > a^2m^2 - b^2$ then straight line $y = mx + c$ is a real chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(ii) If $c^2=a^2m^2-b^2$ then straight line is a tangent to the hyperbola.

(iii) If $c^2<a^2m^2-b^2$ then straight line will be an imaginary chord of the hyperbola.

7.1 CHORD WITH MID POINT

Let (x_1, y_1) be the mid point of a chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then its equation is given by $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$ written as $T = S_1$.

7.2 PARAMETRIC FORM OF CHORD

Let $A(a\sec\theta_1, b\tan\theta_1)$ and $B(a\sec\theta_2, b\tan\theta_2)$ be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then equation of chord AB is given by

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

If $\theta_1 = \theta_2 = \theta$, then this chord become a tangent at point 'θ', equation of which is given by

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

8 ASYMPTOTES

8.1 DEFINITION

A straight line is said to be an asymptote of a hyperbola if it touches the hyperbola at two points at infinity but line does not lie completely on it i.e., the distance of the line from the origin is finite.

Equation of a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point (x_1, y_1) is given by

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \qquad \ldots\text{(i)}$$

and as point lies on the hyperbola we can write

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \qquad \ldots\text{(ii)}$$

Equation (i) with the help of equation (ii) can be written as

$$\frac{x}{a} \pm \frac{y}{b}\sqrt{1 - \frac{a^2}{x_1}} = \frac{a}{x_1} \qquad \ldots\text{(iii)}$$

Now equation (iii) will represent equation of asymptotes if point of tangency is at infinity, so as $x_1 \to \infty$, equation (iii) becomes equation of asymptotes which is given by

$$\frac{x}{a} \pm \frac{y}{b} = 0$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 0 \ , \ \frac{x}{a} - \frac{y}{b} = 0$$

are the two asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Combined equation of the asymptotes as a pair of straight lines can be written as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \qquad \ldots\text{(iv)}$$

From equation (iv) and equation of hyperbola we can say that combined equation of asymptotes of a hyperbola differs by a constant with the equation of hyperbola.

If $S = 0$ represents equation of a hyperbola combined equation of asymptotes of hyperbola can be written as $S + k = 0$ (where k is a constant) and vice-versa. Here value of k can be found by putting $\Delta = 0$ in the combined equation of asymptotes; as it represents two straight lines. Also equation of conjugate hyperbola can be written as $S + 2k = 0$.

8.2 ANGLE BETWEEN ASYMPTOTES

Equation of asymptotes for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given as $\frac{x}{a} - \frac{y}{b} = 0$ and $\frac{x}{a} + \frac{y}{b} = 0$. If θ is the angle included between these two asymptotes then $\theta = 2\tan^{-1}\left(\frac{b}{a}\right)$

8 RECTANGULAR HYPERBOLA

A hyperbola is called rectangular hyperbola if its asymptotes are perpendicular to each other. A hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ will be a rectangular hyperbola if angle between its asymptotes is 90°.

i.e., $\theta = 2\tan^{-1}\frac{b}{a}=\frac{\pi}{2}$

$\Rightarrow$ $b = a$

$\Rightarrow$ Equation of rectangular hyperbola is given by $x^2 - y^2 = a^2$.

Results

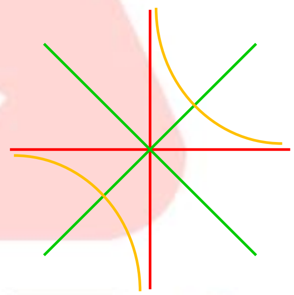
- Any point on it will be of the form $x = a\sec\theta$, $y = a\tan\theta$
- General tangent will be of the form $y = mx \pm a\sqrt{m^2-1}$
- Its director circle will be a point circle i.e. the only pair of tangent which are at right angle are the asymptotes.
- Its normal will be of the form, $x\cos\theta + y\cot\theta = 2a$
- Its asymptotes are $y = x$ and $y = -x$.
- Eccentricity of any rectangular hyperbola is constant and is equal to $\sqrt{2}$.

8.1 ANOTHER FORM OF RECTANGULAR HYPERBOLA

In rectangular hyperbola $x^2 - y^2 = a^2$, we have two pair of mutually perpendicular straight lines.

- Pair of axes of the hyperbola i.e., transverse axis $y = 0$, conjugate axis $x = 0$.
- Pair of asymptotes i.e., $y = x$ and $y = -x$

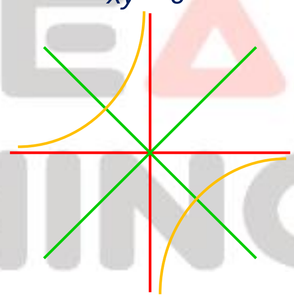
If we interchange the role of these two pairs of straight lines i.e., $x = 0$, $y = 0$ become the pair of asymptotes and $y = x$, becomes the transverse axes and $y = -x$ becomes conjugate axis. Then equation of rectangular hyperbola becomes $xy = c^2$ where $c^2 = \frac{a^2}{2}$.



$xy = c^2$

Remark:

If we make $y = -x$ as transverse axis and $y = x$ as conjugate axis then equation of hyperbola becomes $xy = -c^2$.



$xy = -c^2$

Results:

(i) Any point on rectangular hyperbola $xy = c^2$ will be of the form of $x = ct$, $y = \frac{c}{t}$, which is also the parametric equation of the rectangular hyperbola $xy = c^2$.

(ii) Equation of tangent will be of the form $y = -\frac{1}{t^2}x + \frac{2c}{t}$

Remark:

Here slope of the tangent $m = -\frac{1}{t^2}$; which is always negative for any real t. i.e., tangent to the rectangular hyperbola of the form $xy = c^2$ always make obtuse angle with the positive x-axis. By putting $m = -\frac{1}{t^2}$ we get equation of tangent in the slope form as $y = mx \pm 2c\sqrt{-m}$.

8.2 INTERSECTION OF A CIRCLE AND A RECTANGULAR HYPERBOLA

Consider a circle $x^2 + y^2 + 2gx + 2fy + k = 0$, and rectangular hyperbola $xy = c^2$. By eliminating y from both the equation we get

$$x^4 + 2gx^3 + kx^2 + 2fc^2x + c^4 = 0 \qquad \ldots(i)$$

Roots of equation (i) will give abscissas of the point where circle cuts it.

From equation (i)

$x_1 + x_2 + x_3 + x_4 = -2g$

$\Sigma x_1 x_2 = k$

$\Sigma x_1 x_2 x_3 = -2fc^2$

$x_1 x_2 x_3 x_4 = c^4$

Similarly

If we eliminate x we get

$y_1 + y_2 + y_3 + y_4 = -2f$

$\Sigma y_1 y_2 = k$

$\Sigma y_1 y_2 y_3 = -2gc^2$

$y_1 y_2 y_3 y_4 = c^4$

Here we observe the following:

- The mean point of points of intersection of a rectangular hyperbola and a circle is the mid point of a line segment joining the centres of both the curves.
- The product of abscissas of points of intersection is always equal to product of ordinates of points of intersection and is constant.

MIND MAP

HYPERBOLA

Conditions for

- $ax^2+2hxy+by^2+2gx+2fy+c=0$ to represent a hyperbola:
- $abc+2fgh-af^2-bg^2-ch^2 \neq 0$ and $h^2-ab>0$.
- Further hyperbola is rectangular if $a=b$

Normal to the hyperbola

$x^2/a^2 - y^2/b^2 = 1$

- Normal at (x_1, y_1) to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is $a^2x/x_1 + b^2y/y_1 = a^2+b^2$.
- Normal at $(a\sec\theta, b\tan\theta)$ is $a\cos\theta x + b\cot\theta y = a^2 + b^2$.
- Normal with slope m is

$$y = mx \pm \frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

Various Forms :

- Standard form $x^2/a^2 - y^2/b^2 = 1$
- Here centre of hyperbola (0, 0), vertices $\equiv (\pm a, 0)$ foci $\equiv (\pm ae, 0)$, directrix $x = \pm a/e$, axes $\equiv y = 0, x = 0$ and tangent at the vertices $\equiv x = \pm a$
- Parametric equation $x = a\sec\theta$, $y = b\tan\theta$

Tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$

- Tangent at (x_1,y_1) is $T = 0$
- Tangent with slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$.
- Tangent at $(a\sec\theta, b\tan\theta)$ is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta - 1 = 0$
- The locus of point of intersection of perpendicular tangents is $x^2+y^2 = a^2 - b^2$.
- The combined equation of pair of tangents from point (x_1, y_1) is $T^2 = SS_1$ where $T \equiv xx_1/a^2 - yy_1/b^2 - 1$
- $S = x^2/a^2 - y^2/b^2 - 1$
- $S_1 = x_1^2/a^2 - y_1^2/b^2 - 1$

Conjugate hyperbola and asymptotes

For hyperbola $x^2/a^2 - y^2/b^2 = 1$,

- Conjugate hyperbola is $y^2/b^2 - x^2/a^2 = 1$
- Asymptotes are $y = \pm (b/a)x$

Rectangular hyperbola

- Tangent in parametric form $t^2y + x - 2ct = 0$
- Tangent in slope form $y = mx \pm 2c\sqrt{-m}$
- Normal in parametric form is $ty - t^3x + ct^4 - c = 0$
- Eccentricity is always $\sqrt{2}$.
- Only tangents which are at right angle are asymptotes.

Line and a hyperbola

Line $y = mx + c$ for hyperbola $x^2/a^2 - y^2/b^2 = 1$,

- Is real chord if $c^2 > a^2m^2 - b^2$.
- Is tangent if $c^2 = a^2m^2 - b^2$.
- Is imaginary chord if $c^2 < a^2m^2 - b^2$.
- Equation of chord with mid point as (x_1, y_1) is $T = S_1$.
- Equation of chord of contact from point (x_1, y_1) is $T = 0$