CONTINUITY AND DIFFERENTIABILITY

1 LIMIT OF A FUNCTION

The concept of limit is used to discuss the behaviour of a function close to a certain point.

e.g., $f(x) = \dfrac{x^2 - 1}{x - 1}$

Clearly the function is not defined at $x = 1$, but for values close to $x = 1$ the function can be written as

$f(x) = x + 1$

As x approaches 1 (written as $x \to 1$), $f(x)$ approaches the value 2 (i.e., $f(x) \to 2$) we write this as

$\lim_{x \to 1} f(x) = 2$

It must be noted that it is not necessary for the function to be undefined at the point where limit is calculated. In the above example $\lim_{x \to 2} f(x)$ is the same as the value of function at $x = 2$ i.e., 3.

Informally, we define limit as:

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0, and we write

$\lim_{x \to x_0} f(x) = L$

Sometimes, functions approach different values as x-approaches x_0 from left and right. By left we mean $x < x_0$ and right means $x > x_0$. This is written as $x \to x_0^-$ and $x \to x_0^+$ respectively.

e.g., $f(x) = [x]$ (greatest integer function)

For any integer n,

$\lim_{x \to n^-} f(x) = n - 1$ ...(i)

and $\lim_{x \to n^+} f(x) = n$ ...(ii)

In such cases we say that $\lim_{x \to n} f(x)$ does not exist. The limit in (i) is said to be the left hand limit (L.H.L.) at $x = n$ and that in (ii) is called the right hand limit (R.H.L.) at $x = n$.

2 DEFINITION

2.1 INFORMAL DEFINITION OF LIMITS

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that function approaches the limit L as x approaches x_0, and we write $\lim_{x \to x_0} f(x) = L$.

This definition is "informal" because phrases like arbitrarily close and sufficiently close are imprecise, their meaning depends on the context.

The definition is clear enough and enables us to recognize and evaluate limits of specific function.

2.2 FORMAL DEFINITION OF LIMIT

Let $f(x)$ be defined on an open interval about x_0, except possible at x_0 itself, we say that $f(x)$ approaches the limit L as x approaches x_0 and write $\lim_{x \to x_0} f(x) = L$, if for every number $\in > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \in.$$

2.3 RIGHT HAND LIMIT

If $\lim\limits_{x \to x_0^+} f(x) = L$ for every number $\in > 0$, there exists a corresponding number $\delta > 0$ such that for all x.

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \in$$

Then we call it right hand limit.

e.g. $\lim\limits_{x \to 0^+} \dfrac{x}{|x|} = +1$

2.4 LEFT HAND LIMIT

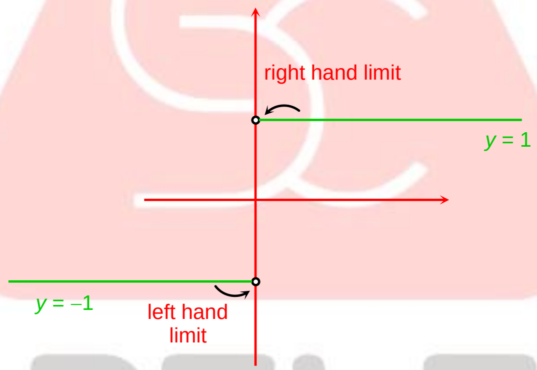
If $\lim\limits_{x \to x_0^-} f(x) = L$ for every number $\in > 0$, there exists a corresponding number $\delta > 0$ such that for all x.

$$x_0 - \delta < x < x_0 \Rightarrow \quad |f(x) - L| < \in$$

Then we call it left hand limit.

e.g. $\lim\limits_{x \to 0^-} \dfrac{x}{|x|} = -1$

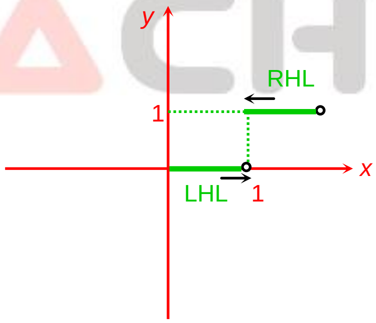
Through graph of $\dfrac{x}{|x|}$ we can easily visualize the things written above.



Now, from the discussions we have just gone through we can easily say that limit of a function will exist iff. LHL and RHL both are finite, unique and equal.

e.g. $\lim\limits_{x \to 1} [x]$ will not exist as LHL = $\lim\limits_{x \to 1^-} [x] = 0$; RHL = $\lim\limits_{x \to 1^+} [x] = 1$

This can be seen graphically as



2.5 ALGEBRA OF LIMITS

(i) $\lim\limits_{x\to a}(c_1f(x)\pm c_2g(x)) = \lim\limits_{x\to a}(c_1f(x)) \pm \lim\limits_{x\to a}(c_2g(x)) = c_1l_1 \pm c_2l_2$,

where c_1 and c_2 are given constants.

(ii) $\lim\limits_{x\to a} f(x).g(x) = \lim\limits_{x\to a} f(x).\lim\limits_{x\to a} g(x) = l_1.l_2$

(iii) $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{l_1}{l_2}$, $\quad l_2 \neq 0$

(iv) $\lim\limits_{x\to a} f(g(x)) = f(\lim\limits_{x\to a} g(x)) = f(l_2)$, if and only if $f(x)$ is continuous at $x = l_2$.

In particular, $\lim\limits_{x\to a}\ln(g(x)) = \ln l_2$ if $l_2 > 0$.

All these theorems must be used with utmost care. For example, we have assumed that l_1 and l_2 are finite. If these are not finite, the given theorems will not be applicable.

e.g. $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$, and if we try to apply the theorems, we get $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$

$= \lim\limits_{x\to 0}\sin x.\lim\limits_{x\to 0}\dfrac{1}{x}$, which does not exist.

Which of course is an absurd result, we are getting this absurd result because in this case the given limit can not be written as the product of two limits as $\lim\limits_{x\to 0}\dfrac{1}{x}$ does not exist.

Similarly, $\lim\limits_{x\to 0}\left[\dfrac{\sin x}{x}\right]$, where [.] denotes the greatest integer function $\neq \left[\lim\limits_{x\to 0}\dfrac{\sin x}{x}\right]$. Here $[x]$ is not continuous at $x = 1$.

(v) $\lim\limits_{x\to a}(1+f(x))^{\frac{1}{g(x)}} = e^{\lim\limits_{x\to a}\frac{\log(1+f(x))}{g(x)}}$

2.6 EVALUATION OF LIMITS

Following are indeterminate forms:

(i) $\dfrac{0}{0}$

(ii) $\dfrac{\infty}{\infty}$

(iii) $0 \times \infty$

(iv) $\infty - \infty$

(v) 0°

(vi) ∞°

(vii) 1^∞

We shall divide the ways of evaluation of limits in five categories:

Simplification: In this method we can use

(i) Direct substitution
(ii) Rationalisation
(iii) Factorization
(iv) Use of formulas like binomial expansion, trigonometric formulas etc.

(i) Direct substitution: We can directly substitute the number at which limit is to be find.

e.g.
- $\lim_{x \to 1} (x^2 + 3x - 2)$ can be find out by this method.

  $\lim_{x \to 1} (x^2 + 3x - 2) = 2$

- $\lim_{x \to -2} |x| = 2$

But before using this method, we have to see that LHL should remain equal to RHL.

e.g. In $\lim_{x \to 1} \sec^{-1} x$, if we directly substitute, we will get $\lim_{x \to 1} \sec^{-1} x$ as 0. But LHL of $\lim_{x \to 1} \sec^{-1} x$ will not exist. So, answer should be, limit does not exist.

(ii) Rationalisation method: Rationalisation is followed when we have powers in fractions on expressions in numerator and denominator or in both. After rationalization, the terms are factorised, which on cancellation give the result.

3 USE OF STANDARD LIMITS

These standard forms are used in case $f(x) \to 0$ when $x \to a$.

(i) $\lim_{x \to a} \dfrac{\sin f(x)}{f(x)} = 1$

(ii) $\lim_{x \to a} \cos f(x) = 1$

(iii) $\lim_{x \to a} \dfrac{\tan f(x)}{f(x)} = 1$

(iv) $\lim_{x \to a} \dfrac{\sin^{-1}(f(x))}{f(x)} = 1$

(v) $\lim_{x \to a} \dfrac{\tan^{-1}(f(x))}{f(x)} = 1$

4 USE OF EXPANSION

(i) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ...$

(ii) $a^x = 1 + \dfrac{x \log a}{1!} + \dfrac{x^2 (\log a)^2}{2!} + ...$ $(a > 0)$

(iii) $\log(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + ...$ $(-1 < x < 1)$

(iv) $(1 + x)^{1/x} = e\left(1 - \dfrac{x}{2} + \dfrac{11x^2}{24} + ...\right)$

5 L' HOSPITAL'S RULE

L′ Hospital's rule is applicable only in $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ indeterminate form. For other forms, first we have to convert them into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form, then we can use it.

It states that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}=......$$

till we are getting $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

Proof: For $\frac{0}{0}$ form: $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$ (as $f(a)$ = 0 and $g(a)$ = 0)

$$=\lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$ and so on

For $\frac{\infty}{\infty}$ form: First put $f(x)=\frac{1}{F(x)}$ and $g(x)=\frac{1}{G(x)}$

Then proceed in same way as $\frac{0}{0}$ form's proof.

6 SANDWICH THEOREM (SQUEEZE PLAY THEOREM)

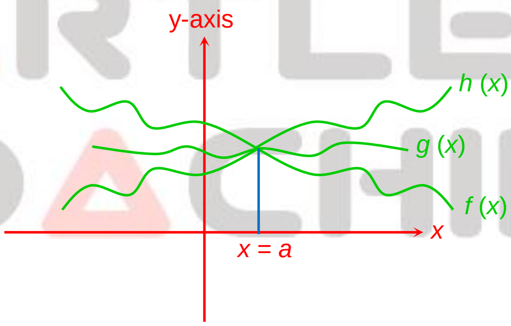
Sandwich theorem helps in calculating the limits, when limits can not be calculated using the above discussed methods.

Sandwich theorem: If $f(x)$, $g(x)$ and $h(x)$ are any three functions such that,

$f(x) \le g(x) \le h(x)\ \forall\ x \in$ neighbourhood of $x = a$.

and $\lim_{x\to a} f(x)=\lim_{x\to a}(h(x))=l$ (say)

Then $\lim_{x\to a} g(x)=l$



7 CONTINUITY

7.1 CONTINUITY OF A FUNCTION

A function $f(x)$ is said to be continuous at $x = a$ if $\lim_{x\to a^-} f(x)=f(a)=\lim_{x\to a^+} f(x)$

i.e. L.H.L = R.H.L. = value of the function at 'a' i.e. $\lim_{x\to a} f(x)=f(a)$.

If $f(x)$ is not continuous a $x = a$, we say that $f(x)$ is discontinuous at $x = a$.

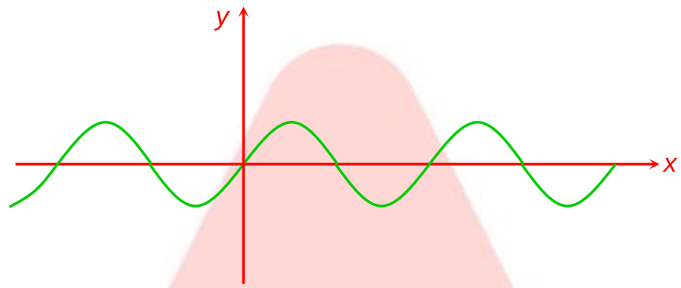
Geometrical meaning of continuity

Function $f(x)$ will be continuous at $x = c$ if there is no break in the graph of function $f(x)$ at the point $(c, f(c))$.

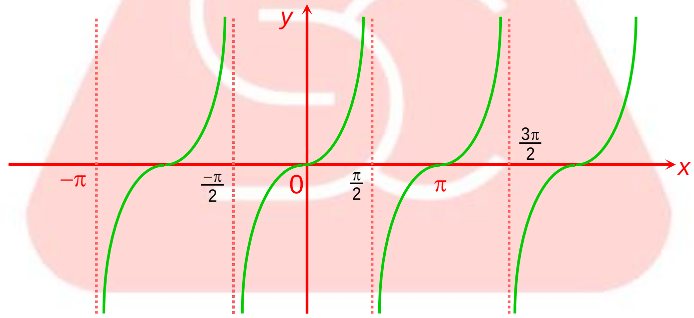
In an interval, function is said to be continuous if there is no break in graph of function in the entire interval.

For example:

- $f(x) = \sin x$ is continuous in its entire domain.



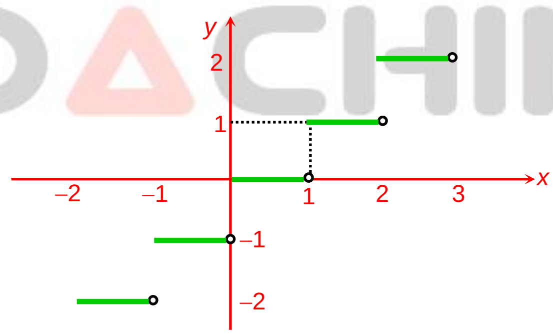
- $f(x) = \tan x$ is discontinuous at $x = (2n+1)\frac{\pi}{2}$ where $n \in \boldsymbol{I}$.



$f(x)$ will be discontinuous at $x = a$, in any of the following cases :

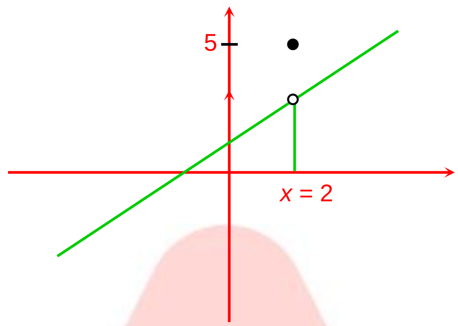
(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal.

For example $y = [x]$ at $x \in \boldsymbol{I}$.



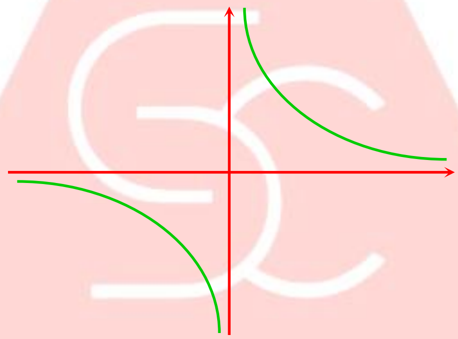
(ii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but not equal to $f(a)$.

For example $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 5 & x = 2 \end{cases}$ at $x = 2$.



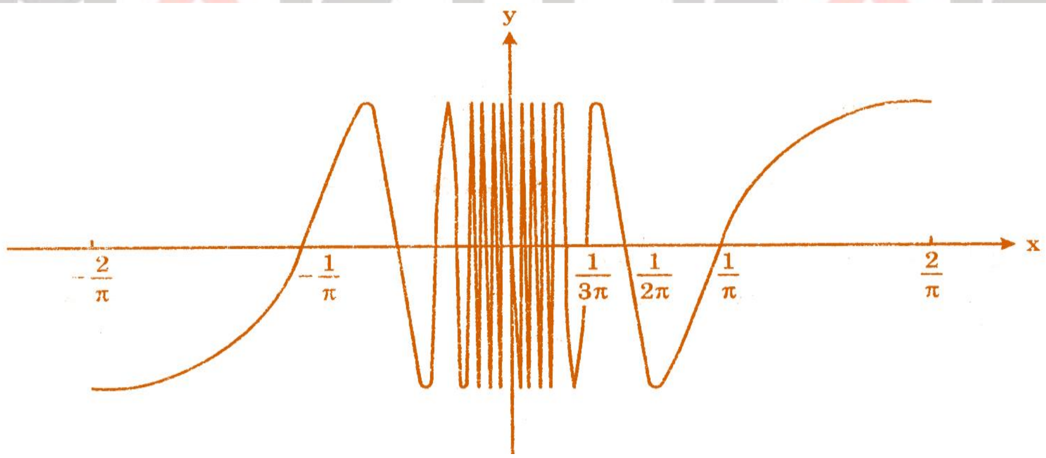
(iii) $f(a)$ is not defined.

For example $y = \dfrac{1}{x}$ at $x = 0$



(iv) At least one of the limits does not exist.

For example $y = \sin\left(\dfrac{1}{x}\right)$ at $x = 0$



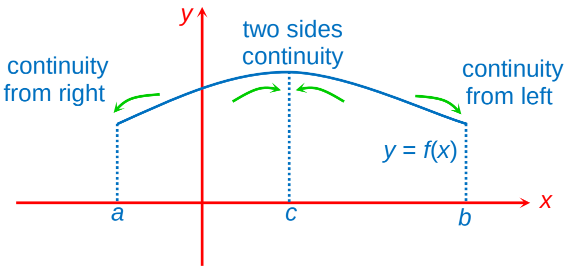
Continuity in an open interval

A function $f(x)$ is said to be continuous in an open interval (a, b) if it is continuous at each point of (a, b).

Continuity in a closed interval

A function $f(x)$ is said to be continuous in an closed interval $[a, b]$ if it is

- continuous at each point (a, b)



Continuity at points *a*, *b* and *c*

- $f(x)$ is continuous from right at $x = a$

  i.e. $\lim_{x \to a^+} f(x) = f(a)$

- $f(x)$ is continuous from left at $x = b$

  i.e. $\lim_{x \to b^-} f(x) = f(b)$

Properties of continuous functions

Let $f(x)$ and $g(x)$ are continuous functions at $x = a$. Then

(i) $c\,f(x)$ is continuous at $x = a$ where c is any constant

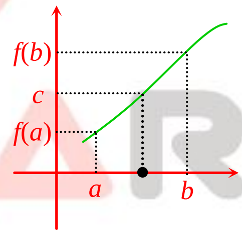
(ii) $f(x) \pm g(x)$ is continuous at $x = a$

(iii) $f(x) \cdot g(x)$ is continuous at $x = a$

(iv) $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$

Intermediate value theorem

If c is any real number between $f(a)$ and $f(b)$, then there exists at least one solution of the equation $f(x) = c$ in the open interval (a, b), if $y = f(x)$ is continuous in the interval.



Types of discontinuities

Basically there are two types of discontinuity:

(i) Removable discontinuity

If $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$, then $f(x)$ has *a* removable discontinuity at $x = a$ and it can be removed by redefining $f(x)$ for $x = a$.

8 DIFFERENTIABILITY

Let $y = f(x)$ be a given function. If at some point, abscissa is x_1 and at other point abscissa is x_2, then it is quite natural then ordinate can be represented by y_1 and y_2 respectively at those points.

$\Delta y = y_2 - y_1$ Δy, represents change in 'y'.

$\Delta x = x_2 - x_1$ Δx, represents change in 'x'.

then $\Delta y = f(x_1 + \Delta x) - f(x_1)$

clearly increment can be positive, negative or may even be zero.

Differential coefficient of $y = f(x)$, with respect to x is defined as the limiting value of $\frac{\Delta y}{\Delta x}$ as Δx tends to zero.

It is usually denoted by $\frac{dy}{dx}$ or $f'(x)$ symbolically.

The derivative of the function with respect to x is the function $f'(x)$ whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. i.e. $\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$

The function is said to be differentiable at $x = a$ if

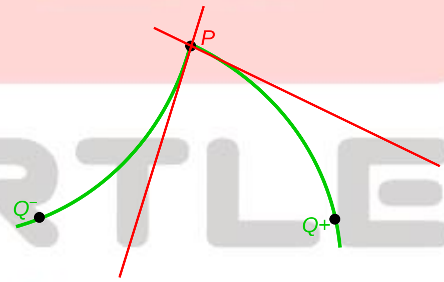
Right hand derivative (RHD) at $x = a$ denoted by $f'(a + 0) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists and

Left hand derivative (LHD) at $x = a$ denoted by $f'(a - 0) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$ also exists.

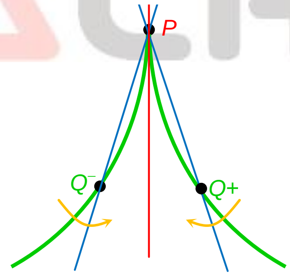
In both these cases, we have assumed $h > 0$.

A function whose graph is otherwise smooth will fail to have a derivative where the graph has

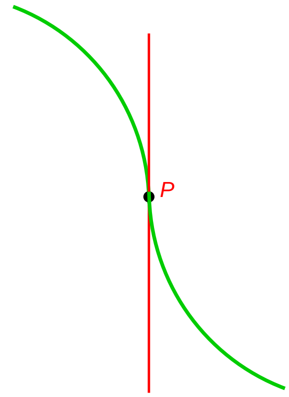
(i) a corner, where the one-sided derivatives differ



(ii) a cusp, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other



(iii) a vertical tangent, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)



(iv) a discontinuity

(a) (b)

A function is continuous at every point where it has a derivative.

Proof: Given that $f'(c)$ exists, we must show that $\lim_{x \to c} f(x) = f(c)$, or, equivalently, that $\lim_{h \to 0} f(c + h) = f(c)$. If $h \neq 0$, then

$$f(c+h) = f(c) + (f(c+h) - f(c))$$

$$= f(c) + \frac{f(c+h) - f(c)}{h}.h\,.$$

Now take limits as $h \to 0$.

$$\lim_{h \to 0} f(c+h) = \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}\,.\,\lim_{h \to 0} h$$

$$= f(c) + f'(c)\,.\,0 = f(c) + 0 = f(c).$$

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$, then f is continuous from that side at $x = c$.

Differentiability in an interval

(i) Differentiability in an open interval (a, b)

The function of $y = f(x)$ is said to be differentiable in (a, b) if it is differentiable at each point $x \in (a, b)$

(ii) In an closed interval $[a, b]$

The function $y = f(x)$, is said to be differentiable in $[a, b]$ if $f'(a + 0)$, $f'(b - 0)$ exist and $f'(x)$ exist for all $x \in (a, b)$.

Properties of differentiability

Let $f(x)$ and $g(x)$ are differentiable functions at $x = a$. Then

(i) $c f(x)$ is differentiable at $x = a$ where c is any constant

(ii) $f(x) \pm g(x)$ is differentiable at $x = a$

(iii) $f(x) \cdot g(x)$ is differentiable at $x = a$

(iv) $f(x)/g(x)$ is differentiable at $x = a$, provided $g(a) \neq 0$

MIND MAP

LIMITS, CONTINUITY AND DIFFERENTIABILITY

Methods to evaluate limits

- **Simplification**
  - (i) Direct substitution
  - (ii) Rationalisation
  - (iii) Factorization
  - (iv) Use of formulas
- **Use of standard limits**

  These standard forms are used in case $f(x) \to 0$ when $x \to a$.

  (i) $\lim_{x\to a} \frac{\sin f(x)}{f(x)} = 1$

  (ii) $\lim_{x\to a} \cos f(x) = 1$

  (iii) $\lim_{x\to a} \frac{\tan f(x)}{f(x)} = 1$

  (iv) $\lim_{x\to a} \frac{\sin^{-1}(f(x))}{f(x)} = 1$

  (v) $\lim_{x\to a} \frac{\tan^{-1}(f(x))}{f(x)} = 1$

  (vi) $\lim_{x\to a} (1+f(x))^{\frac{1}{f(x)}} = e$

  (vii) $\lim_{x\to a} \frac{b^{f(x)}-1}{f(x)} = \log_e b$

  (viii) $\lim_{x\to a} \frac{\log(1+f(x))}{f(x)} = 1$

  (ix) $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$
- **Use of expansions**

  (i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

  (ii) $a^x = 1 + \frac{x\log a}{1!} + \frac{x^2(\log a)^2}{2!} + ...$

  (iii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ....$

  (iv) $(1+x)^{1/x} = e\left(1 - \frac{x}{2} + \frac{11x^2}{24} + ...\right)$

  (v) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.....$

  (vi) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

  (vii) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + ...$

  (viii) $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2 3^2}{5!}x^5....$

  (ix) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + ...$

L′ Hospital's Rule

- It states that $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = ...$

Sandwich theorem

- If $f(x)$, $g(x)$ and $h(x)$ are any three functions such that, $f(x) \le g(x) \le h(x)\ \forall\ x \in$ neighbourhood of $x = a$.

  and $\lim_{x\to a} f(x) = \lim_{x\to a} (h(x)) = l$ (say)

  Then $\lim_{x\to a} g(x) = l$.

Existence of limit

- $\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x)$

Indeterminate forms

- $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞

Algebra of limits

- $\lim_{x\to a} (c_1 f(x) \pm c_2 g(x))$,

  $= \lim_{x\to a}(c_1 f(x)) \pm \lim_{x\to a}(c_2 g(x))$

  where c_1 and c_2 are given constants.
- $\lim_{x\to a} f(x) . g(x) = \lim_{x\to a} f(x) . \lim_{x\to a} g(x)$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l_1}{l_2}$, $l_2 \ne 0$
- $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(l_2)$, if and only if $f(x)$ is continuous at $x = l_2$.

1. Continuity

- $\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = f(a)$

Geometrical meaning

- No break in the graph

2. Differentiability

- $\lim_{h\to 0} \frac{f(x-h)-f(x)}{-h} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Geometrical meaning

- No kinks or sharp corners in graph

Continuity/differentiability in an open interval

A function $f(x)$ is said to be cont./diff. in an open interval (a, b) if it is cont./diff. at each point of (a, b).

Continuity/differentiability in a closed interval

A function $f(x)$ is said to be continuous in an closed interval $[a, b]$ if it is

- Continuous at each point (a, b)
- $f(x)$ is continuous from right at $x = a$ i.e. $\lim_{x\to a^+} f(x) = f(a)$ and

  $f(x)$ is continuous from left at $x = b$ i.e. $\lim_{x\to b^-} f(x) = f(b)$
- The function $y = f(x)$, is said to be differentiable in $[a, b]$ if $f'(a + 0)$, $f'(b - 0)$ exist and $f'(x)$ exist for all $x \in (a, b)$.

Properties of continuity/differentiability

$f(x)$ & $g(x)$ are cont./diff. functions at $x = a$.

- Then $c\,f(x)$ is cont./diff. at $x = a$ where c is any constant
- $f(x) \pm g(x)$ is cont./diff. at $x = a$
- $f(x) . g(x)$ is cont./diff. at $x = a$
- $f(x)/g(x)$ is cont./diff. at $x = a$, provided $g(a) \ne 0$.