1 DETERMINANTS

1.1 EVALUATION OF DETERMINANTS

Determinants of second order: The symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ consisting of 2^2 numbers called elements, arranged in two rows and two columns, is called a determinant of second order. The elements a_1 and b_2 are said to lie along the principal diagonal; the elements a_2 and b_1 are said to lie along the secondary diagonal.

The value of the determinant is obtained by forming the product of the elements along the principal diagonal and subtracting from it the product of the elements along the secondary diagonal.

Thus $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$ ... (i)

Determinants of third order: The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of 3^2 elements arranged in three rows and three columns, is called a determinant of third order. Its value is

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

This may be written as $a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$

or $a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

We can therefore write

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \quad \text{... (ii)}$$

Note that each term of a second order determinant is the product of two quantities and each term of a third order determinant is the product of three quantities.

1.2 MINORS

The minor of a given element of a determinant is the determinant of the elements which remain after deleting the row and the column in which the given element occurs.

For example, the minor of a_1 in (ii) is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, and the minor of b_2 in (ii) is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

The minor of a_1 in (i) is b_2 and b_2 may be considered a determinant of first order. Similarly, the minor of a_2 is b_1.

1.3 COFACTORS

In (ii), the elements a_1, b_1, c_1 are multiplied by

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

These expressions are called the cofactors of the elements a_1, b_1, c_1.

Generally, the cofactor of an element is its minor with its sign or opposite sign prefixed in accordance with the following rule:

For any determinant if a_{ij} be the element at the intersection of the ith row and jth column, then the cofactor of a_{ij} has positive sign or negative sign before minor of a_{ij} according as $i + j$ is even or odd. The determinant may be expanded along any chosen row or column.

The cofactors of the elements a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 will be denoted by A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3 respectively.

For example, element b_3 in (ii) lies at the intersection of the third row and the second column.

Since 3 + 2 = 5 is an odd number, we have

$$B_3 = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The cofactor B_2 of the element b_2 is $+\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$, because element b_2 lies at the intersection of the second row and the second column, and 2 + 2 = 4 is an even number.

Let the determinant (ii) be denoted by Δ. When the cofactors are used, the expansion of the determinant takes the compact form:

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1 = a_2A_2 + b_2B_2 + c_2C_2 = a_3A_3 + b_3B_3 + c_3C_3.$$

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 = b_1B_1 + b_2B_2 + b_3B_3 = c_1C_1 + c_2C_2 + c_3C_3$$

and $a_2A_1 + b_2B_1 + c_2C_1 = 0 = a_2A_3 + b_2B_3 + c_2C_3$ etc.

2 PROPERTIES OF DETERMINANTS

1. If two rows in a determinant are interchanged, the sign of the determinant changes.

For example, by interchanging the two rows of the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, we get the determinant $\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$

But we have $\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

2. If the numbers in one row are added, m times the numbers in another row, the value of the determinant remains unaltered.

For example, $\begin{vmatrix} a_1+ma_2 & b_1+mb_2 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

This rule can be extended to more number of rows for higher order determinants.

3. If rows and columns are interchanged, the value of the determinant remains unaltered.

For example, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

Another way of saying this is that it makes no difference if we reflect the numbers of the determinant in the line of the principal diagonal. This means that any statement that can truly be made about rows in particular results (1) and (2) can equally well be made about columns.

4. If all the numbers in any row are zeros, the value of the determinant is zero.

For example, $\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

5. If two rows are identical, the value of the determinant is zero.

For example, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$

6. If the elements of a row are multiplied by any number m, the determinant is multiplied by m.

For example, $\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

7. Row-Column Operations : The value of determinant remains unchanged when any row (or column) is multiplied by a number or any expression and then added or subtracted from any other row (or column).

i.e. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ma_2 - na_3 & a_2 & a_3 \\ b_1 + mb_2 - nb_3 & b_2 & b_3 \\ c_1 + mc_2 - nc_3 & c_2 & c_3 \end{vmatrix}$

The above operation is written as $C_1 \to C_1 + mC_2 - nC_3$ means C_1 is replaced by $C_1 + mC_2 - nC_3$.

3 SUM OF DETERMINANTS

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ be two third order determinants in which corresponding second and third columns are identical.

Then $\Delta_1 + \Delta_2 = \begin{vmatrix} a_1 + d_1 & b_1 & c_1 \\ a_2 + d_2 & b_2 & c_2 \\ a_3 + d_3 & b_3 & c_3 \end{vmatrix}$

This fact is evident if we expand all the three determinants in terms of column 1 and compare the results.

Similarly if $\Delta_3 = \begin{vmatrix} p_1 & q_1 & r_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

then $\Delta_1 + \Delta_3 = \begin{vmatrix} a_1 + p_1 & b_1 + q_1 & c_1 + r_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Here we note that the corresponding second and third rows are identical.

Similarly the determinant $\begin{vmatrix} d_1 + e_1 + f_1 & d_2 + e_2 + f_2 & d_3 + e_3 + f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Can be decomposed into the sum of three determinants

$$\begin{vmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} e_1 & e_2 & e_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

It may be observed that the determinant

$\begin{vmatrix} a_1 + b_1 & c_1 + d_1 & e_1 + f_1 \\ a_2 + b_2 & c_2 + d_2 & e_2 + f_2 \\ a_3 + b_3 & c_3 + d_3 & e_3 + f_3 \end{vmatrix}$ can be expressed as sum of $2 \times 2 \times 2 = 8$ determinants.

4 MULTIPLICATION OF DETERMINANTS

Two determinants of the same order i.e. each consisting of the same number of rows and equal number of columns can be multiplied to give a determinant of the same order. Thus if A is a 2×2 determinant and *B* is another 2×2 determinant, $A \times B = C$ is also 2×2 determinant. The multiplication is done by a method of working the row of A on the columns of *B*.

The method is as follows:

$$A = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}; B = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}$$

then AB = $\begin{vmatrix} a_1\alpha_1 + a_2\beta_1 & a_1\alpha_2 + a_2\beta_2 \\ b_1\alpha_1 + b_2\beta_1 & b_1\alpha_2 + b_2\beta_2 \end{vmatrix}$

To cite a numerical example for a 3 × 3 determinant; we have

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 6 \\ 3 & 0 & 2 \end{vmatrix} \times \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1\times2+3\times0+4\times1 & 1\times1+3\times1+4\times2 & 1\times2+3\times3+4\times4 \\ 2\times2+(-1)\times0+6\times1 & 2\times1+(-1)\times1+6\times2 & 2\times2+(-1)\times3+6\times4 \\ 3\times2+0\times0+2\times1 & 3\times1+0\times1+2\times2 & 3\times2+0\times3+2\times4 \end{vmatrix}$$

(The first row is obtained by working the first row elements 1, 3, 4 respectively on 2, 0, 1 the first column; then on 1, 1, 2 the 2nd column; then on 2, 3, 4 the 3rd column. Likewise for the 2nd and the 3rd row.)

$$= \begin{vmatrix} 6 & 12 & 27 \\ 10 & 13 & 25 \\ 8 & 7 & 14 \end{vmatrix}$$

Verification

$$A = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 6 \\ 3 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ 0 & -7 & -2 \\ 0 & -9 & -10 \end{vmatrix} \begin{matrix} \\ \to (R_2 - 2R_1) = 70 - 18 = 52 \\ \to (R_5 - 3R_1) \end{matrix}$$

$$B = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -3 & -6 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} \to (R_1 - 2R_3) = -9 + 6 = -3$$

$$C = \begin{vmatrix} 6 & 12 & 27 \\ 10 & 13 & 25 \\ 8 & 7 & 14 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 9 \\ 10 & 13 & 25 \\ 8 & 7 & 14 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 & 9 \\ 0 & 2 & 2 \\ 8 & 7 & 14 \end{vmatrix} \to R_2 - (R_1 + R_3)$$

$$= 3\begin{vmatrix} 2 & 4 & 5 \\ 0 & 2 & 0 \\ 8 & 7 & 7 \end{vmatrix} = 6(14 - 40) = -156$$

$\therefore AB = -156 = C$

The multiplication can also be performed row by row, column by row or column by column.

5 DIFFERENTIATION OF DETERMINANTS

Differentiation of a determinant whose elements are functions of a variable x.

Let $F(x) = \begin{vmatrix} f(x) & g(x) \\ h(x) & u(x) \end{vmatrix}$

Then $F(x) = f(x) \cdot u(x) - g(x) \cdot h(x)$

and $F'(x) = \dfrac{d}{dx} F(x) = \{f(x) \cdot u'(x) + u(x) \cdot f'(x)\} - \{g(x) \cdot h'(x) + h(x)\, g'(x)\}$

$$= \begin{vmatrix} f'(x) & g'(x) \\ h(x) & u(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ h'(x) & u'(x) \end{vmatrix}$$

Thus $F'(x)$ is the sum of two determinants, of which the first one is obtained by differentiating the elements of the first row alone and retaining the second row without any change and the second one is obtained by differentiating the elements of the second row.

Similarly if F(x) = $\begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix}$

then F′(x) = $\begin{vmatrix} f_1'(x) & g_1'(x) & h_1'(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2'(x) & g_2'(x) & h_2'(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3'(x) & g_3'(x) & h_3'(x) \end{vmatrix}$

6 SPECIAL DETERMINANTS

6.1 SYMMETRIC DETERMINANT

If the elements of a determinant are such that $a_{ij} = a_{ji}$, (where a_{ij} is the element of i^{th} row and j^{th} column), then the determinant is said to be a symmetric determinant. The elements situated at equal distances from the diagonal are equal both in magnitude and sign.

e.g.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

6.2 SKEW SYMMETRIC DETERMINANT

If $a_{ij} = -a_{ji}$ (where a_{ij} is the element of i^{th} row and j^{th} column), then the determinant is said to be a skew symmetric determinant, which means that all the diagonal elements are zero and the elements situated at equal distances from the diagonal are equal in magnitude but opposite in sign. The value of a skew symmetric determinant of odd order is zero.

6.3 CIRCULANT DETERMINANTS

The elements of the rows (or columns) are in cyclic arrangement.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc) = -\frac{1}{2}(a+b+c) \times \{(a-b)^2 + (b-c)^2 + (c-a)^2\}$$

7 SYSTEM OF LINEAR EQUATIONS

7.1 THE SYSTEM OF TWO LINEAR EQUATIONS IN TWO UNKNOWNS

Consider the system of two linear equations in two unknowns:

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

Solving the system we get the answer

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} ; \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Note: The given equations are consistent and independent if and only if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$.

7.2 SYSTEM OF THREE EQUATIONS IN TWO UNKNOWNS

The following system of equations

$a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$; $a_3x + b_3y + c_3 = 0$ is consistent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

7.3 THE SYSTEM OF THREE LINEAR EQUATIONS IN THREE UNKNOWNS

Consider the system of three linear equations in three unknowns:

$a_1x + b_1y + c_1z = p$

$a_2x + b_2y + c_2z = q$

$a_3x + b_3y + c_3z = r$

The solution of the system may be expressed as

$x = \frac{d_1}{\Delta}, y = \frac{d_2}{\Delta}, z = \frac{d_3}{\Delta}$, where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ d_1 = \begin{vmatrix} p & b_1 & c_1 \\ q & b_2 & c_2 \\ r & b_3 & c_3 \end{vmatrix},\ d_2 = \begin{vmatrix} a_1 & p & c_1 \\ a_2 & q & c_2 \\ a_3 & r & c_3 \end{vmatrix},\ d_3 = \begin{vmatrix} a_1 & b_1 & p \\ a_2 & b_2 & q \\ a_3 & b_3 & r \end{vmatrix}$$

Note: The determinant Δ is formed by writing the coefficients of x, y, z in order while the determinant appearing in the numerator for any unknown is obtained from Δ by replacing the column of coefficients of that unknown by the column of constants.

7.4 CRAMERS RULE

Consider the system of n linear equations in n unknowns given by

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$.......................................$$
$$.......................................$$
$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

Let $D = \begin{vmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & .... & a_{nn} \end{vmatrix}$

Let D_j be the determinant obtained from D after replacing the j^{th} column by $\begin{vmatrix} b_1 \\ . \\ . \\ . \\ b_n \end{vmatrix}$

Then, if $D \neq 0$, we have $x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, ..., x_n = \frac{D_n}{D}$.

Discuss $D = 0$ cases

Case (i)

If $D = 0$ and the other determinants $D_1 = D_2 = ... = D_n = 0$, then system of equation has infinitely many solutions if all cofactors of $D_1, D_2, ..., D_n$ and D are zero. If any one cofactor of $D_1, D_2, D_3, ..., D_n$ is non zero then system has no solution.

eg. $x + 3y + 2z = 1;\ 2x + 6y + 4z = 5;\ 3x + 9y + 6z = 9$

Here $D_x = D_y = D_z = D = 0$ yet system has no solution where as

$x + 3y + 2z = 1;\ 2x + 6y + 4z = 2;\ 3x + 9y + 6z = 3$ has infinitely many solutions.

Case (ii)

If $D = 0$ but any one of the D_1, D_2 .... or D_n is not equal to zero then the system has no solution, hence is inconsistent.

7.5 THE SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS

Eliminant and non-trivial solution: If the three equations (homogeneous)

$a_1x + b_1y + c_1z = 0; a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$ be considered then there always exists a solution i.e., $x = y = z = 0$. This is called the **Trivial Solution.**

If the three equations are to have a solution other than $x = 0 = y = z$, such a solution is known as **Non-Trivial** solution, then the condition required for the existence of such a solution is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Illustration 17

***Question*:** **Let λ and α be real. Find the set of all values of λ for which the system of linear equations:**

$$\lambda x + (\sin\alpha)\, y + (\cos\alpha)\, z = 0$$
$$x + (\cos\alpha)\, y + (\sin\alpha)\, z = 0$$
$$-x + (\sin\alpha)\, y - (\cos\alpha)\, z = 0$$

has a non-trivial solution. For $\lambda = 1$ find all the values of α.

***Solution*:** The condition for the existence of non-trivial solution (trivial solution is $x = y = z = 0$) is

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0 \quad \text{i.e.,} \quad \begin{vmatrix} 0 & \sin\alpha\,(\lambda+1) & \cos\alpha\,(1-\lambda) \\ 0 & \cos\alpha + \sin\alpha & \sin\alpha - \cos\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

i.e., $(\lambda+1)\sin\alpha\,(\sin\alpha - \cos\alpha) - (1-\lambda)\cos\alpha\,(\cos\alpha + \sin\alpha) = 0$

$$\lambda(\sin^2\alpha + \cos^2\alpha) + \sin^2\alpha - \cos^2\alpha - 2\sin\alpha\cos\alpha = 0$$

$$\lambda = \sin 2\alpha + \cos 2\alpha = \sqrt{2}\sin\left(2\alpha + \frac{\pi}{4}\right)$$

$$-1 \le \frac{\lambda}{\sqrt{2}} \le 1$$

$$-\sqrt{2} \le \lambda \le \sqrt{2}$$

for $\lambda = 1$,

$$\sin\left(2\alpha + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

$$\therefore \quad 2\alpha + \frac{\pi}{4} = \frac{\pi}{4}$$

General solution: $2\alpha + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{4}$

$$2\alpha = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$$

If n is even, $2\alpha = n\pi$

odd, $2\alpha = n\pi - \frac{\pi}{2}$

MIND MAP

DETERMINANTS

DETERMINANT OF ORDER THREE

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

where a_{ij} means element of ith row and jth column.

and $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\,a_{32} - a_{31}\,a_{22}$.

MINOR

Minor of a_{ij} is the determinant of elements which is obtained after deleting ith row and jth column.

COFACTOR

Cofactor of a_{ij} is the product of minor of a_{ij} with $(-1)^{i+j}$

PROPERTIES OF DETERMINANTS

- If two rows/columns in a determinant are interchanged, the sign of the determinant changes
- If the numbers in one row/column are added, m times the number in another row/column, the value of the determinant remains unaltered.
- If rows and columns are interchanged, the value of the determinant remains unaltered.
- If all the numbers in any row/column are zeros, the value of the determinant is zero.
- If two rows/columns are identical, the value of the determinant is zero.
- If the elements of a row/column are multiplied by any number *m*, the determinant is multiplied by *m*.

SPECIAL DETERMINANTS

- **Symmetric determinants:** The elements situated at equal distances from the diagonal are equal both in magnitude and sign, i.e. $a_{ij} = a_{ji}$

  e.g. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

- **Skew Symmetric determinants:** The diagonal elements are zero and the elements situated at equal distances from the diagonal are equal in magnitude but opposite in sign. The value of a skew symmetric determinant of odd order is zero.
- **Circulant determinants:** The elements of the rows (or columns) are in cyclic arrangement.

SUM OF DETERMINANTS

- If each of the elements of a row is expressed as the sum of two numbers, then the determinant may be written as the sum of two determinants.

$$\begin{vmatrix} a_1+a_3 & b_1+b_3 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_3 & b_3 \\ a_2 & b_2 \end{vmatrix}$$

PRODUCT OF DETERMINANTS

- Two determinants can be multiplied together by the following rule.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} p & q \\ r & s \end{vmatrix} = \begin{vmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{vmatrix}$$

SYSTEM OF LINEAR EQUATIONS (CRAMERS RULE)

- Consider the system of *n* linear equations in *n* unknowns given by

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = b_n$$

Let $D = \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & a_{n2} & a_{nn} \end{vmatrix}$

Let D_j be the determinant obtained from D after replacing the j^{th} column by constants.

Then, if $D \neq 0$, we have

$$x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D},\ \ldots\ldots,\ x_n = \frac{D_n}{D}.$$

SYSTEM OF HOMOGENEOUS LINEAR EQUATION

- If the constant b_1 , b_2 …….. b_n are zero then the system of equations have a trivial solution $x_1 = x_2 = \ldots. x_n = 0$.
- It has a non trivial solution also if D is zero.

DIFFERENTIATION OF DETERMINANTS

- F(x) = $\begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix}$ then F′(x) = $\begin{vmatrix} f_1'(x) & g_1'(x) & h_1'(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2'(x) & g_2'(x) & h_2'(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3'(x) & g_3'(x) & h_3'(x) \end{vmatrix}$