DIFFERENTIAL EQUATIONS

1 INTRODUCTION

Generally, any equation, such as

$$f(x, y, a) = 0 \quad \text{.... (i)}$$

represents for each individual value of a, a member of a family of curves. Sometimes it is found necessary to represent the whole family of curves as a single unit and consider them as one for the purpose of studying a common property or characteristic which may run through the members of the family.

From the given equation, solve for a, and the equation $\phi(x,y) = a$ may be obtained; and on differentiating, 'a' gets removed. The resulting equation involving $\frac{dy}{dx}$ is known as a differential equation i.e. the equation representing all the members of the family $f(x, y, a) = 0$ or alternately $\phi(x,y) = a$.

2 DIFFERENTIAL EQUATION

An equation involving an independent variable x, a dependent variable y and the differential coefficients of the dependent variable i.e. $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, ……etc is known as a differential equation. It can also be expressed as a function of variables x, y and derivatives of y w.r.t. x such as

$$f\left(x, y, \frac{dy}{dx}\right) = 0$$

Geometrically, differential equations represent a family of curves having a common property.

3 FORMATION OF DIFFERENTIAL EQUATION

To form a differential equation, we differentiate the given family of curves and eliminate the unknown constants as follows:

(i) Consider the equation $y = ax$. This represents the Cartesian equation to a family of straight lines through the origin.

Differentiating $y = ax$, we get $\frac{dy}{dx} = a$. Eliminating a, we get the differential equation $y = \frac{dy}{dx} \cdot x$.

Hence $y = x\frac{dy}{dx}$ is the differential equation of all straight lines passing through the origin.

(ii) Consider another example, the equation $x^2 + y^2 = a^2$. This, for various a, represents a family of concentric circles with centre at origin.

Differentiating the relation we get

$$2x + 2y\frac{dy}{dx} = 0 \qquad (a \text{ is eliminated})$$

i.e. $$x + y\frac{dy}{dx} = 0$$

which may be said to be the differential equation to a family of concentric circles.

(iii) Now consider another equation representing a family of curves in the form

$$f(x, y, a, b) = 0 \quad \text{.... (i)}$$

containing two arbitrary constants. In this case, since there are two constants, it becomes necessary to differentiate equation twice so that the result contains $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and can be expressed in the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \quad \text{.... (ii)}$$

This equation is said to represent a differential equation of the family of curves represented by equation (i). Thus in the case of $y = ax + b$

$$\frac{dy}{dx} = a$$

$\frac{d^2y}{dx^2} = 0$ which is the differential equation of the family of all straight lines.

4 ORDER AND DEGREE OF DIFFERENTIAL EQUATION

As we know that an equation containing an independent variable, a dependent variable and the derivatives of the dependent variable, is called a differential equation. It has an order and degree defined as follows:

4.1 ORDER OF A DIFFERENTIAL EQUATION

The highest derivative occurring in a differential equation defines its order.

4.2 DEGREE OF A DIFFERENTIAL EQUATION

The power of the highest order derivative occurring in a differential equation is called the degree of the differential equation, for this purpose the differential equation is made free from radicals and fractions of derivatives.

4.3 EXAMPLES

Differential equation	Order of D.E.	Degree of D.E.
• $\frac{dy}{dx} + 4y = \sin x$	1	1
• $\left(\frac{d^2y}{dx^2}\right)^4 + \left(\frac{dy}{dx}\right)^5 - y = e^x$	2	4
• $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3y = \cos x$	2	1
• $\frac{dy}{dx} = \frac{x^4 - y^4}{xy(x^2 + y^2)}$	1	1
• $y = x\frac{dy}{dx} + \sqrt{a^2\left(\frac{dy}{dx}\right)^2 + b^2}$		
$\Leftrightarrow (x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (y^2 - b^2) = 0$	1	2
• $\frac{d^2y}{dx^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}$		
$\Leftrightarrow \left(\frac{d^2y}{dx^2}\right)^2 - \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = 0$	2	2

5 SOLUTION OF A DIFFERENTIAL EQUATION

5.1 EQUATIONS WITH SEPARABLE VARIABLE

Differential equations of the form

$$\frac{dy}{dx} = f(x, y)$$

can be reduced to form

$$\frac{dy}{dx} = g(x)\, h(y)$$

where it is possible to take all terms involving x and dx on one side and all terms involving y and dy to the other side, thus separating the variables and integrating.

5.2 EQUATIONS REDUCIBLE TO EQUATIONS WITH SEPARABLE VARIABLE

A differential equations of the form

$$\frac{dy}{dx} = f(ax + by + c)$$

can not be solved by separating the variables directly. By substituting $ax + by + c = t$ and $a + b\frac{dy}{dx} = \frac{dt}{dx}$, the differential equation can be separated in terms of variables x and t.

5.3 HOMOGENEOUS DIFFERENTIAL EQUATIONS

A differential equations of the form

$$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$$

where $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of x and y of the same degree, is called a homogeneous equation.

It can also be written in form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, by dividing both the functions by x^n where n is the degree of function.

To solve this equation, substitute

$$\frac{y}{x} = t \quad \text{or} \quad y = tx$$

$$\Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$$

Then the equation reduces to $t + x\frac{dt}{dx} = f(t)$ which can be easily reduced to variable separable as $\frac{dt}{f(t) - t} = \frac{dx}{x}$.

5.4 EQUATIONS REDUCIBLE TO HOMOGENEOUS EQUATION

A differential equation of the form $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$

where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, can be reduced to homogeneous equation by putting $x = X + h$ and $y = Y + k$. where h, k are such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$ also

$$\frac{dy}{dx} = \frac{dY}{dX}$$

hence equation reduces to $\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$ (homogeneous form).

If $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ = λ, then $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}=\frac{\lambda(a_1x+b_1y)+c_1}{(a_1x+b_1y)+c_2}$ can be solved by putting $a_1x + b_1y = t$, as then it reduces to equation with variable separable.

5.5 LINEAR DIFFERENTIAL EQUATIONS

A differential equation of the form

$$\frac{dy}{dx}+P(x)\,y=Q(x)$$

where $P(x)$ and $Q(x)$ are functions of x only or constants, is known as linear differential equation.
To solve this equation, we try to convert both sides as perfect differentials multiplying the equation by another function of x say $R(x)$.

Then $R(x)\frac{dy}{dx}+P(x)\,R(x)y=Q(x)\,R(x)$

This can be reduced to $\frac{d}{dx}(y\,R(x))=Q(x)\,R(x)$

if $\frac{d}{dx}(R(x))=P(x)\,R(x)$

$\Rightarrow \quad P(x)=\frac{R'(x)}{R(x)}$

On integrating both sides.

$\Rightarrow \quad \int P(x)\,dx=\log R(x)$

$\Rightarrow \quad R(x) = e^{\int P(x)\,dx}$

This function is known as integrating factor, I.F. = $e^{\int Pdx}$.
The solution of differential equation is given by

$$y\,(I.F.)=\int Q(x)\,(I.F)\,dx$$

5.6 EQUATIONS REDUCIBLE TO THE LINEAR DIFFERENTIAL EQUATION

(i) If equation is of the form.

$R(y)\frac{dy}{dx}+P(x)\,S(y)=Q(x)$ such that $\frac{dS}{dy}=R$, then put $S(y) = t$

$\Rightarrow \quad \frac{dt}{dx}=\frac{dS}{dx}=\frac{dS}{dy}\cdot\frac{dy}{dx}=\frac{Rdy}{dx}$

Thus differential equation reduces to $\frac{dt}{dx}+P(x)t=Q(x)$

which is linear differential equation.

(ii) Bernoulli's equation

Differential equation of the form

$$\frac{dy}{dx}+Py=Qy^n\ ;$$

P, Q are functions of x is called Bernoulli's equation.
To solve this, divide the equation by y^n, then

$\frac{1}{y^n}\frac{dy}{dx}+P\frac{1}{y^{n-1}}=Q \qquad$ Put $\quad \frac{1}{y^{n-1}}=t$

$\Rightarrow \quad -\frac{(n-1)}{y^n}\frac{dy}{dx}=\frac{dt}{dx}$

Differential equation reduces to

$$\frac{dt}{dx}+\left(\frac{1}{n-1}\right)P(x)t=\frac{Q(x)}{(1-n)}$$

(iii) Sometimes the differential equation can be written as linear differential equation in *x* as dependent variable and *y* as independent variable.

5.7 EXACT DIFFERENTIAL EQUATIONS

These are equations of the form $\sum f(\phi(x,y))\,d\phi=0$

Sometimes given differential equations may be expressed as different derivatives of functions, and then integrating them gives the result.

Following results may be helpful in such problems:

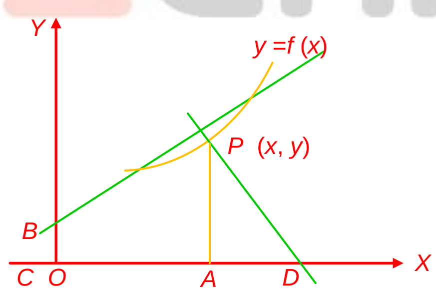
- $d(xy)=xdy+ydx$
- $d\left(\frac{x}{y}\right)=\frac{ydx-xdy}{y^2}$
- $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$
- $d\left(\frac{x^2}{y}\right)=\frac{2xydx-x^2dy}{y^2}$
- $d\left(\frac{y^2}{x}\right)=\frac{2xydy-y^2dx}{x^2}$
- $d\left(\frac{x^2}{y^2}\right)=\frac{2xy^2dx-2x^2ydy}{y^4}$
- $d\left(\frac{y^2}{x^2}\right)=\frac{2x^2ydy-2xy^2dx}{x^4}$
- $d\left(\tan^{-1}\frac{x}{y}\right)=\frac{ydx-xdy}{x^2+y^2}$
- $d\left(\tan^{-1}\frac{y}{x}\right)=\frac{xdy-ydx}{x^2+y^2}$
- $d\,[\ln(xy)]=\frac{xdy+ydx}{xy}$
- $d\left(\ln\left(\frac{x}{y}\right)\right)=\frac{ydx-xdy}{xy}$
- $d\left[\frac{1}{2}\ln(x^2+y^2)\right]=\frac{xdx+ydy}{x^2+y^2}$
- $d\left[\ln\left(\frac{y}{x}\right)\right]=\frac{xdy-ydx}{xy}$
- $d\left(-\frac{1}{xy}\right)=\frac{xdy+ydx}{x^2y^2}$
- $d\left(\frac{e^x}{y}\right)=\frac{ye^xdx-e^xdy}{y^2}$
- $d\left(\frac{e^y}{x}\right)=\frac{xe^ydy-e^ydx}{x^2}$
- $d\,(x^my^n)=x^{m-1}y^{n-1}(mydx+nxdy)$.

5.8 TO SOLVE DIFFERENTIAL EQUATION OF THE FIRST ORDER BUT OF HIGHER DEGREE.

In such differential equations we substitute the lower degree derivative by some other variable.

6 GEOMETRICAL APPLICATION

Some differential equations are formed due to some results on tangents and normals as follows:



- The equation of the tangents at $P(x, y)$ to the curve $y = f(x)$ is $Y-y=\frac{dy}{dx}(X-x)$.

- The equation of the normal at $P(x, y)$ to the curve $y = f(x)$ is $Y - y = \dfrac{-1}{dy/dx}(x - y)$.
- The length of the tangent = $CP = |y|\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}$.
- The length of the normal = $PD = |y|\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$.
- The length of the Cartesian subtangent = $CA = \left|y\dfrac{dx}{dy}\right|$.
- The length of the Cartesian subnormal = $AD = \left|y\dfrac{dy}{dx}\right|$.
- The initial ordinate of the tangent = $OB = y - x\dfrac{dy}{dx}$.

7 ORTHOGONAL TRAJECTORIES

The orthogonal trajectories of a family of curves form another family of curves such that each curve of one family cuts all the curves of the other family at right angles.

The differential equation of the orthogonal trajectories of the curves $f\left(x, y, \dfrac{dy}{dx}\right) = 0$ is the family of curves whose differential equation is $f\left(x, y, \dfrac{-dx}{dy}\right) = 0$.

Method: To find the orthogonal trajectories of a family of curves whose differential equation is known, put $-\dfrac{dx}{dy}$ in place of $\dfrac{dy}{dx}$ in the equation. The resulting differential equation is the equation of the orthogonal trajectories.

Note: If the orthogonal trajectories form the same family of curves as the given family of curves then the given system of curves is called self-orthogonal.

8 OTHER APPLICATIONS

Differential equations, their formation and solution is of practical importance. Following are some examples of application of differential equations in problems involving rate of change of temperature and rate of change of flow of fluids etc.

MIND MAP

DIFFERENTIAL EQUATIONS

DIFFERENTIAL EQUATION

- An equation containing an independent variable, a dependent variable and the derivatives of the dependent variable, is called a differential equation.

ORDER OF DIFFERENTIAL EQUATION

- The highest derivative occurring in a differential equation defines its order.

DEGREE OF DIFFERENTIAL EQUATION

- The power of the highest order derivative occurring in a differential equation is called the degree of the differential equation, for this purpose the differential equation is made free from radicals and fractions of

ORTHOGONAL TRAJECTORIES

- The orthogonal trajectories of a family of curves form another family of curves such that each curve of one family cuts all the curves of the other family at right angles.
- The differential equation of the orthogonal trajectories of the curves $f\left(x, y, \frac{dy}{dx}\right) = 0$ is the family of curves whose differential equation is $f\left(x, y, \frac{-dx}{dy}\right) = 0$.

GEOMETRICAL APPLICATION

- Tangents at $P(x, y)$ to $y = f(x)$ is $Y - y = \frac{dy}{dx}(X - x)$.
- Normal at $P(x, y)$ to $y = f(x)$ is $Y - y = \frac{-1}{dy/dx}(X - x)$.

The length of the

- Tangent $= CP = |y|\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$.
- Normal $= PD = |y|\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.
- Subtangent $= CA = \left|y\frac{dx}{dy}\right|$.
- Subnormal $= AD = \left|y\frac{dy}{dx}\right|$.
- The initial ordinate of the tangent $= OB = y - x\frac{dy}{dx}$.

EQUATIONS WITH SEPARABLE VARIABLE

- Differential equations of the form $\frac{dy}{dx} = f(x, y)$ can be reduced to form $\frac{dy}{dx} = g(x)\,h(y)$ where it is possible to take all terms involving x and dx on one side and all terms involving y and dy to the other side, thus separating the variables and integrating.

EQUATIONS REDUCIBLE TO EQUATIONS WITH SEPARABLE VARIABLE

- A differential equations of the form $\frac{dy}{dx} = f(ax + by + c)$ can not be solved by separating the variables directly. By substituting $ax + by + c = t$ and $a + b\frac{dy}{dx} = \frac{dt}{dx}$, the differential equation can be separated in terms of variables x and t.

HOMOGENEOUS DIFFERENTIAL EQUATIONS

- A differential equations of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ where $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of x and y of the same degree, is called a homogeneous equation. It can also be written in form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, by dividing both the functions by x^n where n is the degree of function. To solve this equation, substitute $y/x = t$

EQUATIONS REDUCIBLE TO HOMOGENEOUS EQUATION

- A differential equations of the form $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ can be reduced to homogeneous equation by putting $x = X + h$ and $y = Y + k$. where h, k are such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$

LINEAR DIFFERENTIAL EQUATIONS

- A differential equations of the form $\frac{dy}{dx} + P(x)\,y = Q(x)$ where $P(x)$ and $Q(x)$ are functions of x only or constants, is known as linear differential equation. I.F. $= e^{\int P dx}$. The solution of differential equation is given by $y\,(I.F.) = \int Q(x)\,(I.F)\,dx$

EQUATIONS REDUCIBLE TO THE LINEAR DIFFERENTIAL EQUATION

- If equation is of the form. $R(y)\frac{dy}{dx} + P(x)\,S(y) = Q(x)$ such that $\frac{dS}{dy} = R$ then put $S(y) = t$