INTEGRALS

1 INTEGRAL AS ANTI-DERIVATIVE

If $f(x)$ is a differentiable function such that $f'(x) = g(x)$, then integration of $g(x)$ w.r.t. x is $f(x) + c$. Symbolically it is written as $\int g(x)\,dx = f(x) + c$

here c is known as constant of integration and it can take any real value. For example $\frac{d}{dx}(\tan x) = \sec^2 x$, so, $\int \sec^2 x\,dx = \tan x + c$

2 LIST FOR STANDARD FORMULAE

Based upon the about method and the previous knowledge of differentiation of standard functions, here is the list of integration of standard functions.

Functions $f(x)$ (Integrand)	Integration $\int f(x)\,dx$
constant k	$kx + c$
x^n	$\frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$
$1/x \quad (x \neq 0)$	$\ln\lvert x\rvert + c$
$a^x \quad (a > 0)$	$a^x/\ln a + c$
e^x	$e^x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sec^2 x$	$\tan x + c$
$\text{cosec}^2 x$	$-\cot x + c$
$\sec x \tan x$	$\sec x + c$
$\text{cosec} x \cot x$	$-\text{cosec} x + c$

$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x + c$
$1/(1+x^2)$	$\tan^{-1}x + c$
$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\sec^{-1}x + c$

Theorem1.

$$\frac{d}{dx}\left\{\int f(x)\,dx\right\} = f(x)$$

Proof: Let $\int f(x)\,dx = F(x)$ ...(i)

Then, $\frac{d}{dx}\{F(x)\} = f(x)$

$\therefore \quad \frac{d}{dx}\left\{\int f(x)\,dx\right\} = f(x)$

Theorem 2.

Two integrals of the same function can differ only by a constant.

Proof:

Let $f_1(x)$ and $f_2(x)$ be two integrals of $g(x)$. Then by definition $f_1'(x) = g(x)$ and

$f_2'(x) = g(x)$ for all possible values of real x.

$\Rightarrow$ $f_1'(x) = f_2'(x) \quad \forall\, x \in R$

Let $h(x) = f_1(x) - f_2(x)$

$\Rightarrow$ $h'(x) = 0 \quad \forall\, x \in R$

Now consider the interval $[a, b]$ $(a < b)$, then by Lagrange's Mean value's theorem, there exists some $c \in (a, b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a}$$

Since $h'(x) = 0\ \forall\, x \in R$ so $h'(c) = 0$

$\Rightarrow$ $h(b) = h(a)$

$\Rightarrow$ $h(x)$ is a constant function

Let $h(x) = c$

$\Rightarrow$ $f_1(x) - f_2(x) = c$

Hence two integrals of the same function can differ only by a constant.

Theorem 3.

(i) $\int (af(x) + bg(x))dx$

$= a\int f(x)dx + b\int g(x)dx$

where a and b are constants

(ii) $\int f(x)dx = g(x) + c$

then $\int f(ax+b)\,dx = \frac{1}{a}g(ax+b) + c$

where a and b are constants and $a \neq 0$

3 INTEGRATION USING TRIGONOMETRIC IDENTITIES

When the integrand consists of trigonometric functions, we use known identities to convert it into a form which can easily be integrated. Some of the identities useful for this purpose are given below:

(i) $2\sin^2\left(\frac{x}{2}\right) = (1 - \cos x)$

(ii) $2\cos^2\left(\frac{x}{2}\right) = (1 + \cos x)$

(iii) $2\sin a \cos b = \sin(a+b) + \sin(a-b)$

(iv) $2\cos a \sin b = \sin(a+b) - \sin(a-b)$

(v) $2\cos a \cos b = \cos(a+b) + \cos(a-b)$

(vi) $2\sin a \sin b = \cos(a-b) - \cos(a+b)$

4 INTEGRATION BY SUBSTITUTION

It is not always possible to find the integral of a complicated function only by observation, so we need some methods of integration and integration by substitution is one of them.

This methods has 3 parts:

(i) Direct substitution

(ii) Standard substitution

(iii) Indirect substitution

4.1 DIRECT SUBSTITUTION

If $\int f(x)\,dx = g(x) + c$ then in $I = \int f(h(x))\,h'(x)\,dx$,

we put $h(x) = t$ $\Rightarrow$ $h'(x)\,dx = dt$

so $I = \int f(t)dt = g(t) + c = g(h(x)) + c$

4.2 STANDARD SUBSTITUTION

In some standard integrand or a part of it, we have standard substitution. List of standard substitution is as follows:

Integrand	Substitution
$x^2 + a^2$ or $\sqrt{x^2+a^2}$	$x = a\tan\theta$
$x^2 - a^2$ or $\sqrt{x^2-a^2}$	$x = a\sec\theta$
$a^2 - x^2$ or $\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a+x}$ and $\sqrt{a-x}$	$x = a\cos2\theta$
$(x \pm \sqrt{x^2 \pm a^2})^n$	expression inside the bracket = t
$\frac{2x}{a^2-x^2}, \frac{2x}{a^2+x^2}, \frac{a^2-x^2}{a^2+x^2}$	$x = a\tan\theta$
$2x^2 - 1$	$x = \cos\theta$
$\frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}}$ $(n \in N, n > 1)$	$\frac{x+a}{x+b} = t$

4.3 INDIRECT SUBSTITUTION

If integrand $f(x)$ can be rewritten as product of two functions. $f(x) = f_1(x)\,f_2(x)$, where $f_2(x)$ is a function of integral of $f_1(x)$, then put integral of $f_1(x) = t$.

5 INTEGRATION BY PARTS

If integrand can be expressed as product of two functions, then we use the following formula.

$$\int f_1(x)\,f_2(x)\,dx = f_1(x)\int f_2(x)dx - \int\left((f_1'(x)\int f_2(x)\,dx)\right)dx$$

where $f_1(x)$ and $f_2(x)$ are known as first and second function respectively.

Remarks:

(i) We do not put constant of integration in 1st integral, we put this only once in the end.

(ii) Order of $f_1(x)$ and $f_2(x)$ is normally decided by the rule ILATE, where I $\rightarrow$ inverse, L $\rightarrow$ Logarithms, A $\rightarrow$ Algebric, T $\rightarrow$ trigonometric and E $\rightarrow$ exponential.

5.1 SPECIAL USE OF INTEGRATION BY PARTS

(i) $\int f(x)\,dx = \int (f(x))\,.\,1\,dx$

Now integrate taking $f(x)$ as 1st function and 1 as 2nd function. In most of the cases, $f(x)$ is an inverse or logarithmic function.

(ii) $\int \frac{f(x)}{[g(x)]^n}\,dx = \int \frac{f(x)}{g'(x)} \cdot \frac{g'(x)}{[g(x)]^n}\,dx$

Now integrate taking $\frac{f(x)}{g'(x)}$ as 1st function and $\frac{g'(x)}{[g(x)]^n}$ as 2nd function.

(iii) If integrand is of the form $e^x f(x)$, then rewrite $f(x)$ as sum of two functions in which one is derivative of other.

$$\int e^x f(x)\,dx = \int e^x (g(x) + g'(x))\,dx$$

$$= e^x g(x) + c$$

6 INTEGRATION USING PARTIAL FRACTIONS

When integrand is a rational function i.e. of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are the polynomials functions of x, we use the method of partial fraction. For example we can rewrite $\frac{1}{(3x-1)(3x+2)}$ as $\frac{1}{3(3x-1)} - \frac{1}{3(3x+2)}$.

If the degree of $f(x)$ is less then degree of $g(x)$ and $g(x) = (x-a_1)^{\alpha_1}$ …… $(x^2+b_1x+c_1)^{\beta_1}$ ……. then we can put

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_1)^2} + \ldots\ldots + \frac{A_{\alpha_1}}{(x-a_1)^{\alpha_1}} \ldots\ldots\ldots\ldots$$

$$+ \frac{B_1x+C_1}{(x^2+b_1x+c_1)} + \frac{B_2x+C_2}{(x^2+b_1x+c_1)^2} + \ldots\ldots + \frac{B_{\beta_1}x+C_{\beta_1}}{(x^2+b_1x+c_1)^{\beta_1}} \ldots\ldots$$

Here A_1, A_2……., A_{α_1} …….. , B_1 , B_2 ……. , B_{β_1} …… C_1, C_2……. C_{β_1} …….. are the real constants and these can be calculated by reducing both sides of the above equation as identity in polynomial form and then by comparing the coefficients of like powers. The constants can also be obtained by putting some suitable numerical values of x in both sides of the identity.

If degree of $f(x)$ is more than or equal to degree of $g(x)$, then divide $f(x)$ by $g(x)$ so that the remainder has degree less than of $g(x)$.

7 ALGEBRAIC INTEGRALS

Using the technique of standard substitution and integration by parts, we can derive the following formulae.

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left|x + \sqrt{x^2 + a^2}\right| + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + c$$

$$\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \ln\left|x + \sqrt{x^2 + a^2}\right| + c$$

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln\left|x + \sqrt{x^2 - a^2}\right| + c$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + c$$

7.1 INTEGRAL OF THE FORM

$$\int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \sqrt{ax^2 + bx + c}\ dx$$

Here in each case write $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$ put $x + \frac{b}{2a} = t$ and use the standard formulae.

7.2 INTEGRALS OF THE FORM

$$\int \frac{(ax+b)\,dx}{\sqrt{cx^2 + ex + f}}, \int \frac{(ax+b)\,dx}{cx^2 + ex + f}, \int (ax+b)\sqrt{cx^2 + ex + f}\ dx$$

Here write $ax + b = A(2cx + e) + B$
Find A and B by comparing, the coefficients of x and constant term.

7.3 INTEGRALS OF THE FORM

$$\int \frac{(ax^2 + bx + c)\,dx}{\sqrt{(ex^2 + fx + g)}}, \int \frac{(ax^2 + bx + c)\,dx}{(ex^2 + fx + g)}, \int (ax^2 + bx + c)\sqrt{(ex^2 + fx + g)}\,dx$$

Here put $ax^2 + bx + c = A(ex^2 + fx + g) + B(2ex + f) + c$

find the values of A, B and C by comparing the coefficient of x^2, x and constant term.

Here, put $cx + d = t^2$

7.5 INTEGRALS OF THE FORM

$$\int \frac{dx}{(ax+b)\sqrt{ex^2 + fx + g}}$$

Here put $ax + b = \frac{1}{t}$

7.6 INTEGRALS OF THE FORM

$$\int \frac{(ax+b)\,dx}{(cx+e)\sqrt{ex^2+fx+g}}$$

Here put $(ax + b) = A\,(cx + e) + B$

Find the values of A and B by comparing the coefficients of x and constant term.

7.7 INTEGRALS OF THE FORM

$$\int \frac{(ax^2+bx+c)\,dx}{(ex+f)\sqrt{gx^2+hx+i}}$$

Here put $ax^2 + bx + c = A(ex + f)\,(2gx + h) + B(ex + f) + C$

Find the values of A , B and C by comparing the coefficients of x^2, x and constant term.

7.8 INTEGRALS OF THE FORM

$$\int \frac{x\,dx}{(ax^2+b)\sqrt{(cx^2+e)}}$$

Here put $cx^2 + e = t^2$

7.9 INTEGRALS OF THE FORM

$$\int \frac{dx}{(ax^2+b)\sqrt{(cx^2+e)}}$$

Here 1st put $x = \frac{1}{t}$ and then take the expression inside the square root as y^2

7.10 INTEGRALS OF THE TYPE

$$\int x^m\,(a + bx^n)^p\,dx\ (p \neq 0).$$

Here we have the following cases.

Case I: If p is a natural number, then expand $(a + bx^n)^p$ by binomial theorem and integrate.

Case II: If p is a negative integer and m and n are rational number, then put $x = t^k$, when k is the LCM of denominators of m and n.

Case III: If $\frac{m+1}{n}$ is an integer and p is rational number, put $(a + bx^n) = t^k$, when k is the denominator of p.

Case IV: If $\frac{m+1}{n} + p$ is an integer, put $\frac{a+bx^n}{x^n} = t^k$,

where k is the denominator of p.

8 TRIGONOMETRIC INTEGRALS

8.1 INTEGRALS OF THE FORM

$$\int \left(\frac{f(\sin x, \cos x)}{g(\sin x, \cos x)}\right) dx = \int R(\sin x, \cos x)\,dx$$

where f and g both are polynomials in $\sin x$ and $\cos x$. Here we can convert them in algebraic by putting $\tan\frac{x}{2} = t$ after writing $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$.

Now, put $\tan\frac{x}{2} = t$ and $\sec^2\frac{x}{2}dx = 2dt$.

The new integrand can now be easily integrated.

Some time instead of putting the above substitution we go for below procedure.

(i) If $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, put $\cos x = t$

(ii) If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, put $\sin x = t$

(iii) If $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ put $\tan x = t$, after dividing numerator and denominator by $\cos^2 x$.

8.2 INTEGRALS OF THE FORM

$$\int\left(\frac{p\sin x + q\cos x + r}{a\sin x + b\cos x + c}\right)dx$$

Here put $p\sin x + q\cos x + r = A(a\sin x + b\cos x + c) + B(a\cos x - b\sin x) + C$

Values of A, B and C can be obtained by comparing the coefficients of $\sin x$, $\cos x$ and constant term by this technique. The given integral becomes sum of 3 integrals in which 1st two are very easy and in 3rd we can put $\tan\frac{x}{2} = t$.

8.3INTEGRALS OF THE FORM

$$\int \sin^p x \cos^q x \, dx$$

When any one or both of p and q are odd integers, then put $\cos x = t$ or $\sin x = t$ and when p and q are rational number such that $\frac{p+q-2}{2}$ is a negative integer, then put $\tan x = t$ or $\cot x = t$.

9 SUCCESSIVE REDUCTION IN INTEGRATION

Sometimes the integrand is a function of x as well as of n($n \in \mathbf{N}$), then we use this method.

MIND MAP

INDEFINITE INTEGRATION

Substitution

(i) If $\int f(x)\,dx = g(x) + c$, then

$$\int f(h(x))h'(x)\,dx = g(h(x)) + c$$

(ii) in $\sqrt{x^2 + a^2}$ put $x = a\tan\theta$

in $\sqrt{x^2 - a^2}$ put $x = a\sec\theta$

in $\sqrt{a^2 - x^2}$ put $x = a\sin\theta$

in $\sqrt{a+x}, \sqrt{a-x}$ put $x = a\cos\theta$

in $(x \pm \sqrt{x^2 \pm a^2})^n$ put $x \pm \sqrt{x^2 \pm a^2} = t$

(iii) If $f_1(x)$ is a function of integral of $f_2(x)$ then in $\int f_1(x) f_2(x)\,dx$ put integral of $f_2(x)$

Integration as anti derivative

If $f'(x) = g(x)$, then

$$\int g(x)\,dx = f(x) + c$$

Partial function

$$\frac{f(x)}{(x-a)^n} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \ldots\ldots + \frac{A_n}{(x-a)^n}$$

$$\frac{g(x)}{(x^2+ax+b)^m} = \frac{A_1x + B}{(x^2+ax+b)} + \ldots. + \frac{A_mx + B_m}{(x^2+ax+b)^m}$$

Algebraic integrals

(i) in $\sqrt{ax^2 + bx + c}$

make the perfect square

(ii) in $\int \frac{(ax+b)\,dx}{cx^2 + ex + f}$

write $ax + b = A(2cx + e) + B$

(iii) in $\int \frac{(ax^2 + bx + c)\,dx}{\sqrt{ex^2 + fx + g}}$

write $ax^2 + bx + c$

$= A(ex^2 + fx + g) + B(2ex + f) + C$

(iv) in $\int \frac{dx}{(ax+b)\sqrt{(cx^2 + ex + f)}}$

Put $ax + b = 1/t$

(v) in $\int \frac{(ax+b)\,dx}{(cx+e)\sqrt{fx^2 + gx + h}}$

Put $ax + b = A(cx + e) + B$

(vi) in $\int \frac{(ax^2 + bx + c)\,dx}{(ex+f)\sqrt{gx^2 + hx + i}}$

Put $ax^2 + bx + c$

$= A(gx^2 + hx + i) + B(2gx + h)(ex + f) + C$

(vii) in $\int \frac{x\,dx}{(ax^2+b)\sqrt{(cx^2+e)}}$

Put $cx^2 + e = t^2$

(viii) in $\int \frac{dx}{(ax^2+b)\sqrt{cx+d}}, \int \frac{dx}{(ax+b)\sqrt{cx+d}}$

Put $cx + d = t^2$

By partial fraction

$$\int f_1(x) f_2(x)\,dx$$

$$= f_1(x)\int f_2(x)\,dx - \int\left(f_1'(x)\int f_2(x)\,dx\right)dx$$

and order of $f_1(x)$ and $f_2(x)$ are normally decided by ILATE.

Trigonometric integration

(i) in $R(\sin x, \cos x)$

If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

Put $\sin x = t$

If $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

Put $\cos x = t$

If $R(-\sin x, -\cos x) = R(\sin x, \cos x)$

Put $\tan x = t$

(ii) in $\int \frac{(a\sin x + b\cos x + c)\,dx}{e\sin x + f\cos x + g}$

Put $a\sin x + b\cos x + c$

$= A(e\sin x + f\cos x + g)$

$+ B(e\cos x - f\sin x) + C$