Limits & Derivatives

1 LIMIT OF A FUNCTION

The concept of limit is used to discuss the behaviour of a function close to a certain point.

e.g., $f(x) = \dfrac{x^2 - 1}{x - 1}$

Clearly the function is not defined at $x = 1$, but for values close to $x = 1$ the function can be written as

$f(x) = x + 1$

As x approaches 1 (written as $x \to 1$), $f(x)$ approaches the value 2 (i.e., $f(x) \to 2$) we write this as

$\lim_{x \to 1} f(x) = 2$

It must be noted that it is not necessary for the function to be undefined at the point where limit is calculated. In the above example $\lim_{x \to 2} f(x)$ is the same as the value of function at $x = 2$ i.e., 3.

Informally, we define limit as:

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0, and we write

$\lim_{x \to x_0} f(x) = L$

Sometimes, functions approach different values as x-approaches x_0 from left and right. By left we mean $x < x_0$ and right means $x > x_0$. This is written as $x \to x_0^-$ and $x \to x_0^+$ respectively.

e.g., $f(x) = [x]$ (greatest integer function)

For any integer n,

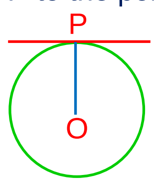
$\lim_{x \to n^-} f(x) = n - 1$ ...(i)

and $\lim_{x \to n^+} f(x) = n$ ...(ii)

In such cases we say that $\lim_{x \to n} f(x)$ does not exist. The limit in (i) is said to be the left hand limit (L.H.L.) at $x = n$ and that in (ii) is called the right hand limit (R.H.L.) at $x = n$.

1.1 TANGENT LINE AND SLOPE PREDICTOR

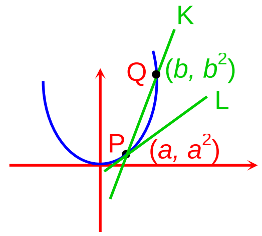
In elementary geometry the line tangent to a circle at a point P is defined as the straight line through P that is perpendicular to the radius OP to the point P.



A general graph $y = f(x)$ has no radius for as to use, but the line tangent to the graph at the point P should be the straight line through P that has in some sense the same direction at P as the curve itself. Because a line's "direction" is determined by its slope.

Let us take an example:

(a) Determine the slope of the line L tangent to the parabola $y = x^2$ at the point (a, a^2).



We can't immediately calculate the slope *L*, because we know the co-ordinate of only one point $\left(a,\ a^2\right)$ of *L*. Hence we begin with another line whose slope we can compute.

In figure secant line *K* that passes through the point *P* and a very close point $Q\left(b,\ b^2\right)$ of the parabola $y = x^2$. Let us write $h = \Delta x = b - a$ for the difference of *x*-co-ordinate of *P* and *Q*. (the notation Δx is an increment, or change in the value of *x*). The co-ordinates of *Q* is given by

$$b = a + h \text{ and } b^2 = (a+h)^2$$

Hence the difference in *y*-co-ordinates of *P* and *Q* is

$$\Delta y = b^2 - a^2 = (a+h)^2 - a^2$$

Because *P* and *Q* are two different points, we can use the definition of slope to calculate the slope m_{PQ} of the secant line *K* through *P* and *Q*. If you change the value of *h* = Δ*x*, then also change the line *K* and thereby change its slope. Therefore m_{PQ} depends on *h*.

$$m_{PQ} = \frac{\Delta y}{\Delta x}$$

$$= \frac{(a+h)^2 - a^2}{(a+h) - a} = \left(\frac{2ah + h^2}{h}\right) = \frac{h(2a+h)}{h}$$

because *h* is non-zero we can cancel it out and it becomes $= (2a+h)$

Now, if *Q* moves along the curve closer and closer to the point *P*. The line *K* still passes through *P* and *Q*. It is very close to *P* then *h* approaches zero, the secant line *K* comes closer to coinciding with the tangent line *L*.

Out idea is to define the tangent line *L* as the limiting position of the secant line *K*.

h: approaches zero
Q: approaches *P*, and so
K: approaches *L*, mean while

the slope of *K* approaches the slope of *L*.

As the number *h* approaches zero, what values does the slope $m_{PQ} = 2a + h$ approaches we can state this question of the "limiting value" of 2*a* + *h* by writing

$$\lim_{h \to 0} (2a + h)$$

Here "lim" is an abbreviation for the word "limit" and *h* → 0 is an abbreviation for the phrase "*h* approaches zero", then we can give the answer that what is the limit of 2*a* + *h* as *h* approaches zero.

For any specific value of *a* we can investigate this question numerically by calculating values of $2a + h$ with values of *h* that become closer and closer to zero, such as the values *h* = –0.1, *h* = –0.01, *h* = –0.001, *h* = –0.0001, ……, or the values *h* = 0.5, *h* = 0.1, *h* = 0.05, *h* = 0.01, …… For instance, the tables of values (in figure) indicate that with *a* = 2 and *a* = –4, we should conclude that

$$\lim_{h \to 0} (2 + h) = 2 \text{ and } \lim_{h \to 0} (-4 + h) = -4$$

More generally, it seems clear from the table in figure that

$$\lim_{h \to 0} m_{PQ} = \lim_{h \to 0} (2a + h) = 2a \qquad \ldots\text{(i)}$$

h	–4+*h*
0.5	–3.5

h	$2+h$
0.1	2.1
0.01	2.01
0.001	2.001
0.0001	2.0001
↓	↓
0	2

h	$2a+h$
0.01	$2a$ + 0.01
0.001	$2a$ + 0.001
!	!
↓	↓
0	$2a$

As $h \to 0$ (first column), $2 + h$ approaches 2 (second column)	As $h \to 0$ (first column), $-4 + h$ approaches -4 (second column)	As $h \to 0$ (first column), $2a + h$ approaches $2a$ (second column)

This, finally, answers our original question: The slope $m = m_{PQ}$ of the line tangent to the parabola $y = x^2$ at the point (a, a^2) is given by

$$m = 2a \qquad \ldots\text{(ii)}$$

The formula in equation (ii) is a “slope predictor” for (lines tangent to) the parabola $y = x^2$. Once we know the slope of the line tangent to the curve at a given point of the curve, we can then use the point-slope formula to write an equation of this tangent line.

We define the slope m of the line tangent to the graph $y = f(x)$ at the point $P(a, f(a))$ to be

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If $(a+h)$ is x $(h = x - a)$. We set that x approaches a as h approaches 0.

So, $m = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)$

2 THE LIMIT LAWS

2.1 CONSTANT LAW

If $f(x) \equiv C$, where C is a constant [so $f(x)$ is a constant function], then

$$\lim_{x \to a} f(x) = \lim_{x \to a} C = C \qquad \ldots\text{(i)}$$

2.2. SUM LAW

If both of the limits

$$\lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M$$

exist, then

$$\lim_{x \to a} [f(x) \pm g(x)] = \left[\lim_{x \to a} f(x)\right] \pm \left[\lim_{x \to a} g(x)\right] = L \pm M \qquad \ldots\text{(ii)}$$

(The limit of a sum is the sum of the limits, the limit of a difference is the difference of the limits).

2.3 PRODUCT LAW

If both of the limits

$$\lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M$$

exist, then

$$\lim_{x \to a} [f(x) g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right] = LM \qquad \ldots\text{(iii)}$$

(The limit of a product is the product of the limits).

2.4 QUOTIENT LAW

If both of the limits

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M$$

exist and if $M \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \quad \text{...(iv)}$$

(The limit of a quotient is the quotient of the limits, provided that the limit of the denominator is not zero.)

2.5 ROOT LAW

If n is a positive integer and if $a > 0$ for even values of n, then

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a} \quad \text{...(v)}$$

The case $n = 1$ of the root law is obvious

$$\lim_{x\to a} x = a$$

Examples (ii) and (iii) show how the limit laws can be used to evaluate limits of polynomials and rational functions.

Example 1:

$$\lim_{x\to 3} \left(x^2 + 2x + 4\right) = \left(\lim_{x\to 3} x^2\right) + \left(\lim_{x\to 3} 2x\right) + \left(\lim_{x\to 3} 4\right)$$

$$= \left(\lim_{x\to 3} x\right)^2 + 2\left(\lim_{x\to 3} x\right) + \left(\lim_{x\to 3} 4\right) = 3^2 + 2.3 + 4 = 19$$

Example 2:

$$\lim_{x\to 3} \frac{2x+5}{x^2+2x+4} = \frac{\lim_{x\to 3}(2x+5)}{\lim_{x\to 3}\left(x^2+2x+4\right)}$$

$$= \frac{2.3+5}{3^2+2.3+4} = \frac{11}{19}$$

Note: In examples 1 and 2, we systematically applied the limit laws until we could simply substitute 3 for $\lim_{x\to 3} x$ at the final step. To determine the limit of a quotient of polynomials, we must verify before this final step that the limit of the denominator is not zero. If the denominator limit is zero, then the limit may fail to exist.

2.6 SOME IMPORTANT FORMULAE

(i) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(ii) $\lim_{x\to 0} \frac{\tan x}{x} = 1$

(iii) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

(iv) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$

(v) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(vi) $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$

(vii) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(viii) $\lim_{x\to 0} (1+x)^{1/x} = e$

(ix) $\lim_{x\to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(x) $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$

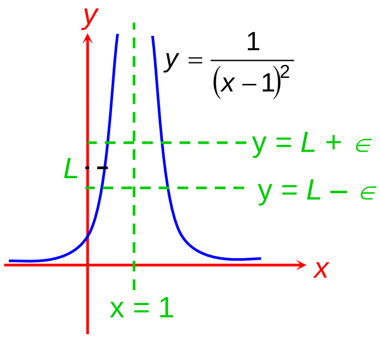
Illustration 1

Question: **Investigate $\lim_{x\to 1} \frac{1}{(x-1)^2}$..**

Solution: Because $\lim_{x\to 1}(x-1)^2 = 0$, we cannot apply the quotient law.

Moreover, we can make $\frac{1}{(x-1)^2}$ arbitrarily large be choosing *x* sufficiently close to 1.

Hence $\frac{1}{(x-1)^2}$ can not approach any (finite) number *L* as *x* approaches 1.



Therefore, the limit in this example does not exist. You can see the geometric reason if you examine the graph of $y = \frac{1}{(x-1)^2}$ in figure.

As $x \to 1$, the corresponding point (*x, y*) ascends the curve near the vertical line *x* = 1. It must therefore leave the indicated strip between the two horizontal lines $x = L - \in$ and $x = L + \in$ that bracket the proposed limit *L*. Thus, the point (*x, y*) cannot approach the point (1, *L*) as $x \to 1$.

Illustration 2

Question: **Investigate** $\lim_{x\to 2}\frac{x^2-4}{x^2+x-6}$.

Solution: We cannot immediately apply the quotient law because the denominator approaches zero as *x* approaches 2.

If the numerator were approaching some number other than zero as $x \to 2$, then the limit would fail to exist (as in Illustration 1). But here the numerator does approach zero, so there is a possibility that a factor of the numerator can be canceled with the same factor of the denominator, thus removing the zero-denominator problem. Indeed,

$$\lim_{x\to 2}\frac{x^2-4}{x^2+x-6} = \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x+3)} = \lim_{x\to 2}\frac{x+2}{x+3} = \frac{4}{5}$$

We can cancel the factor *x* – 2 because it is nonzero: $x \neq 2$ when we evaluate the limit as *x* approaches 2. Moreover, this verifies the numerical limit of 0.8 that we found in given limit.

3 LIMITS OF TRIGONOMETRIC FUNCTIONS

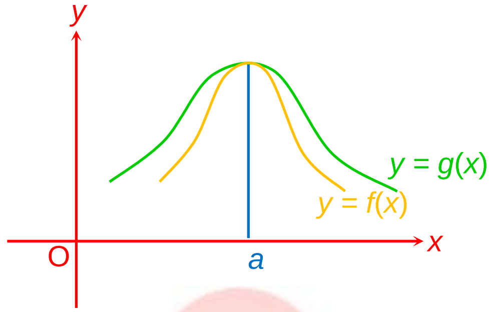
The following facts about functions in general come in handy in calculating limits of some trigonometric functions.

Theorem 1

Let *f* and *g* be two real valued functions with the same domain such that $f(x) \le g(x)$ for all *x* in the domain of definition.

For some *a*, if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

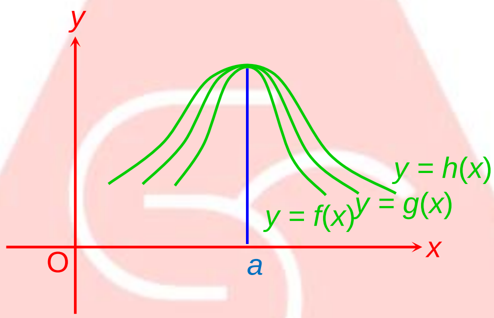
This is illustrated in figure:



Theorem 2 (Sandwich Theorem)

Let *f*, *g* and *h* be real functions such that $f(x) \le g(x) \le h(x)$ for all *x* in the common domain of definition. For some real member *a*, if $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x) = l$.

This is illustrated in figure:

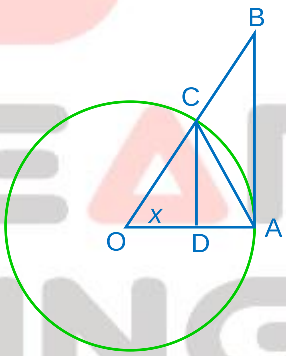


Given below is a geometric proof of the following important inequality relating trigonometric functions.

$$\cos x < \frac{\sin x}{x} < 1 \text{ for } 0 < |x| < \frac{\pi}{2} \qquad \ldots\text{(i)}$$

Proof:

We know that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$. Hence, it is sufficient to prove the inequality for $0 < x < \frac{\pi}{2}$. In the figure is the centre of the unit circle such that the angle *AOC* is *x* radians and $0 < x < \frac{\pi}{2}$. Lines segments *BA* and *CD* are perpendiculars to *OA*. Further, join *AC*. Then



Area of ΔOAC < Area of sector OAC < Area of ΔOAB

i.e. $\frac{1}{2} OA.CD < \frac{x}{2\pi}.\pi.(OA)^2 < \frac{1}{2} OA.AB$

i.e. $CD < x.OA < AB$

From ΔOCD

$\sin x = \frac{CD}{OA}$ (since $OC = OA$) and hence $CD = OA \sin x$

Also, $\tan x = \frac{AB}{OA}$ and $AB = OA.\tan x$

Thus, $OA \sin x < OA.x < OA.\tan x$

Since length *OA* is positive, we have $\sin x < x < \tan x$

Since $0 < x < \frac{\pi}{2}$, $\sin x$ is positive and thus by dividing throughout by sin *x*, we have

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

Taking reciprocals throughout, we have $\cos x < \frac{\sin x}{x} < 1$, which complete the proof.

Now, $\lim_{x\to 0} \cos x < \lim_{x\to 0} \frac{\sin x}{x} < \lim_{x\to 0} 1 \Rightarrow \lim_{x\to 0} \frac{\sin x}{x} = 1$

Illustration 3

Question: **Evaluate $\lim_{x\to 2} \frac{x-3}{x+4}$.**

Solution: $\lim_{x\to 2} \frac{x-3}{x+4} = \frac{2-3}{2+4} = -\frac{1}{6}$

Illustration 4

Question: **Evaluate $\lim_{x\to 2} \frac{x^5-32}{x^3-8}$.**

Solution: $\lim_{x\to 2} \frac{x^5-32}{x^3-8} = \lim_{x\to 2}\left(\frac{x^5-32}{x-2}\right) \div \left(\frac{x^3-8}{x-2}\right) = \lim_{x\to 2}\left(\frac{x^5-2^5}{x-2}\right) \div \lim_{x\to 2}\left(\frac{x^3-2^3}{x-2}\right) = 5(2)^4 \div 3(2)^2 = \frac{20}{3}$

$\left[\text{As } \lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$

Illustration 5

Question: **Evaluate $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$.**

Solution: $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$, we rationalize i.e. multiply and divide by $\left(\sqrt{1+x}+\sqrt{1-x}\right)$

$= \lim_{x\to 0} \frac{(1+x)-(1-x)}{x[\sqrt{1+x}+\sqrt{1-x}]} = \lim_{x\to 0} \frac{2x}{x[\sqrt{1+x}+\sqrt{1-x}]}$

$= \lim_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} = \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{1+1} = \frac{2}{2} = 1$

Illustration 6

Question: **If $\lim_{x\to 2} \frac{x^n-2^n}{x-2} = 80$ and $n \in I^+$, find n.**

Solution: $\lim_{x\to 2} \frac{x^n-2^n}{x-2} = 80 \Rightarrow n.2^{n-1} = 5.2^4 \Rightarrow n = 5$

Illustration 7

Question: **Evaluate $\lim_{x\to 0} \frac{\cos ax - \cos bx}{x^2}$.**

Solution: $\lim_{x\to 0} \frac{\cos ax - \cos bx}{x^2} = \lim_{x\to 0} \frac{2\sin\frac{ax+bx}{2}.\sin\frac{bx-ax}{2}}{x^2}$

$$= \lim_{x\to 0} 2.\frac{\sin\frac{(a+b)x}{2}}{\frac{(a+b)x}{2}}\times\frac{a+b}{2}.\frac{\sin\frac{(b-a)x}{2}}{\frac{(b-a)x}{2}}\times\frac{(b-a)}{2} = 2\times\frac{a+b}{2}\times\frac{b-a}{2} = \frac{b^2-a^2}{2}$$

Illustration 8

Question: Evaluate $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-\frac{\pi}{4}}$.

Solution: $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-\frac{\pi}{4}}$

Put $x-\frac{\pi}{4}=t$, as $x\to\frac{\pi}{4}$, $t\to 0$

$$= \lim_{t\to 0}\frac{\sin\left(\frac{\pi}{4}+t\right)-\cos\left(\frac{\pi}{4}+t\right)}{t} = \lim_{t\to 0}\frac{\sin\frac{\pi}{4}\cos t+\cos\frac{\pi}{4}\sin t-\cos\frac{\pi}{4}\cos t+\sin\frac{\pi}{4}\sin t}{t}$$

$$= \lim_{t\to 0}\frac{\frac{1}{\sqrt{2}}\cos t+\frac{1}{\sqrt{2}}\sin t-\frac{1}{\sqrt{2}}\cos t+\frac{1}{\sqrt{2}}\sin t}{t}$$

$$= \frac{2}{\sqrt{2}}\lim_{t\to 0}\frac{\sin t}{t} = \sqrt{2}\times 1 = \sqrt{2}$$

4 DERIVATIVES

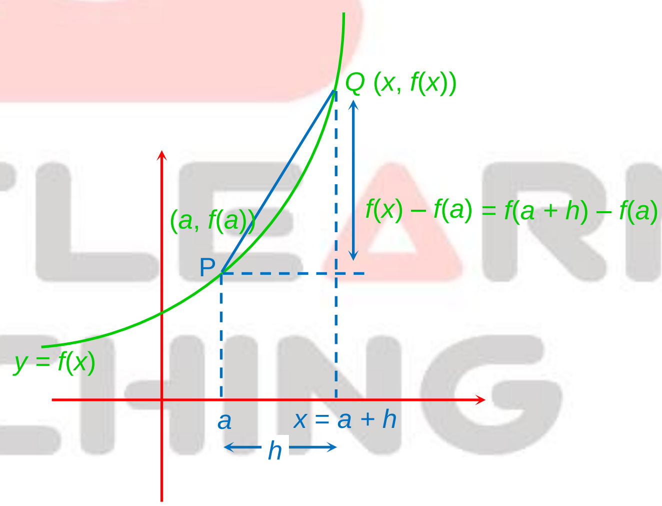
We know that the tangent line to the curve $y=f(x)$ at the point $(a, f(a))$ has slope

$$m = m(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Provided that this limit exists. From this, we get a new function f' the derivative of the original function *f*.

The derivative of the function *f* is the function f' define by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$



for all *x* for which this limit exist.

In the figure $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Note: Suppose *f* is a real valued function, the function defined by

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

wherever the limit exists is defined to be the derivative of *f* at *x* and is denoted by $f'(x)$. This definition of derivative is also called the first principle of derivative.

Thus $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Clearly the domain of definition of $f'(x)$ is wherever the above limit exists. There are different notations for derivative of a function. Sometimes $f'(x)$ is denoted by $\frac{d}{dx}[f(x)]$ or if $y = f(x)$, it is denoted by $\frac{dy}{dx}$. This is referred to as derivative of $f(x)$ or y with respect to x. It is also denoted by $D[f(x)]$. Further, derivative of f at $x = a$ is also denoted by $\left.\frac{d}{dx}f(x)\right|_a$ or $\left.\frac{df}{dx}\right|_a$ or even $\left(\frac{df}{dx}\right)_{x=a}$.

Illustration 9

Question: **Find the derivative of $f(x) = \frac{1}{x}$.**

Solution: We have $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{\frac{1}{(x+h)} - \frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x-(x+h)}{x(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-h}{x(x+h)}\right]$$

$$= = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

Illustration 10

Question: **Differentiate $e^{\sqrt{x}}$, with respect to x, from first principle.**

Solution: Let $f(x) = e^{\sqrt{x}}$

$\therefore$ $f'(x) = \lim_{h\to 0} \frac{e^{\sqrt{x+h}} - e^{\sqrt{x}}}{h}$, h is small increment in x

$$= \lim_{h\to 0} \frac{e^{\sqrt{x}}[e^{\sqrt{x+h}-\sqrt{x}} - 1]}{h}$$

$$= \lim_{h\to 0} e^{\sqrt{x}} . \frac{e^{\sqrt{x+h}-\sqrt{x}} - 1}{\sqrt{x+h}-\sqrt{x}} . \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= e^{\sqrt{x}}.1. \lim_{h\to 0} \frac{x+h-x}{h[\sqrt{x+h}+\sqrt{x}]}$$

$$= e^{\sqrt{x}}. \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

5 DERIVATIVES OF SOME OF THE FREQUENTLY USED FUNCTIONS

Function	Derivative
c (constant)	0
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{\lvert x \rvert\sqrt{x^2-1}}$
$\operatorname{cosec}^{-1} x$	$\frac{-1}{\lvert x \rvert\sqrt{x^2-1}}$
$\log_e x$	$1/x$
$\log_a x$	$\frac{1}{x\log_e a}$
x^n	nx^{n-1}
a^x	$a^x \log_e a$
e^x	e^x

The above written derivatives can be easily found by using the definition of differentiation.

6 RULES TO FIND OUT DERIVATIVES

Let u and v are differentiable functions of 'x'.

(i) The sum rule

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

e.g. $$\frac{d}{dx}(2e^x + 3\log x) = 2\frac{de^x}{dx} + 3\frac{d(\log x)}{dx} \qquad = 2e^x + \frac{3}{x}$$

Illustration 11

Question: Differentiate $5\sin x - 2\log_e x$.

Solution: $$\frac{d}{dx}(5\sin x - 2\log_e x) = \frac{d}{dx}(5\sin x) - \frac{d}{dx}(2\log_e x) = 5\cos x - \frac{2}{x}$$

(ii) Product rule

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

e.g. $\dfrac{d\,((\sin x)\,e^x)}{dx} = \sin x\,\dfrac{de^x}{dx} + e^x\,\dfrac{d\,(\sin x)}{dx}$ = $(\sin x)\,e^x + (\cos x)\,e^x$.

Illustration 12

Question: **Differentiate $x^2e^x\sin x$.**

Solution: First we differentiate x^2e^x

$$\frac{d}{dx}\left(x^2e^x\right) = x^2\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(x^2\right) = x^2e^x + 2xe^x$$

Now, $\dfrac{d}{dx}\left(x^2e^x\sin x\right) = x^2e^x\dfrac{d}{dx}(\sin x) + \sin x\dfrac{d}{dx}\left(x^2e^x\right)$

$$= x^2e^x\cos x + \sin x\left(x^2 + 2x\right)e^x$$

$$= e^x\left(x^2\cos x + x^2\sin x + 2x\sin x\right) = xe^x\left(x\cos x + x\sin x + 2\sin x\right)$$

(iii) The quotient rule

Here $v(x) \neq 0$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\dfrac{v\,du}{dx} - \dfrac{u\,dv}{dx}}{v^2}$$

e.g. $$\frac{d}{dx}\left(\frac{\tan x}{x}\right) = \frac{x\dfrac{d(\tan x)}{dx} - (\tan x)\dfrac{dx}{dx}}{x^2}$$

$$= \frac{x\sec^2 x - \tan x}{x^2}$$

Illustration 13

Question: **Differentiate $\dfrac{e^x}{1+\sin x}$.**

Solution: $$\frac{d}{dx}\left(\frac{e^x}{1+\sin x}\right) = \frac{(1+\sin x)\dfrac{d}{dx}\left(e^x\right) - e^x\dfrac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)e^x - e^x\cos x}{(1+\sin x)^2} = \frac{e^x(1+\sin x-\cos x)}{(1+\sin x)^2}$$

(iv) Chain rule

The chain rule is probably the most widely used differentiation rule in mathematics. Chain rule says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points.

The formula is $[f(g(x))]' = f'(g(x)).g'(x)$

Illustration 14

Question: **Differentiate $\sin x^2$.**

Solution: Put $y = x^2$ and $z = \sin y$

Then $\dfrac{dy}{dx} = 2x$ and $\dfrac{dz}{dy} = \cos y$

$$\therefore\ \frac{d}{dx}\left(\sin x^2\right) = \frac{dz}{dx} = \frac{dz}{dy}.\frac{dy}{dx} = (\cos y)\,(2x) = \left(\cos x^2\right)(2x) = 2x\cos x^2$$

This solution can be rewritten using a more convenient notation in the following manner:

$$\frac{d}{dx}\left(\sin x^2\right) = \frac{d\left(\sin x^2\right)}{d\left(x^2\right)}.\frac{d\left(x^2\right)}{dx} = \cos x^2.2x = 2x\cos x^2$$

Illustration 15

Question: Differentiate $\sin^3 x . \sin 3x$ w.r.t. x.

Solution:

$$\frac{d}{dx}\left(\sin^3 x.\sin 3x\right) = \sin^3 x.\frac{d}{dx}(\sin 3x) + \sin 3x.\frac{d}{dx}(\sin x)^3$$

$$= \sin^3 x.\cos 3x.3 + \sin 3x.3(\sin x)^2.\cos x$$

$$= 3\sin^2 x[\sin x\cos 3x + \cos x.\sin 3x]$$

$$= 3\sin^2 x.\sin(x+3x) = 3\sin^2 x\sin 4x$$

Illustration 16

Question: Differentiate $\sin\left(\frac{2x}{1+x^2}\right) + e^{\sqrt{1+x^2}}$ with respect to x.

Solution: Let $y = \sin\left(\frac{2x}{1+x^2}\right) + e^{\sqrt{1+x^2}}$

$$\frac{dy}{dx} = \cos\left(\frac{2x}{1+x^2}\right)\left(\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)\right) + e^{\sqrt{1+x^2}}\left(\frac{d}{dx}\sqrt{1+x^2}\right)$$

$$\frac{dy}{dx} = \frac{2\left(1+3x^2\right)}{\left(1+x^2\right)^2}\cos\left(\frac{2x}{1+x^2}\right) + \frac{x}{\sqrt{1+x^2}}e^{\sqrt{1+x^2}}$$

Illustration 17

Question: Differentiate $\frac{x^3\sqrt{5+x}}{(7-3x)^5\sqrt[3]{8+5x}}$ with respect to x.

Solution: Let $y = \frac{x^3\sqrt{5+x}}{(7-3x)^5\sqrt[3]{8+5x}}$, take log on both sides

$$\log y = 3\log x + \frac{1}{2}\log(5+x) - 5\log(7-3x) - \frac{1}{3}\log(8+5x)$$

Differentiate w.r.t. x

$$\frac{1}{y}.\frac{dy}{dx} = 3.\frac{1}{x} + \frac{1}{2}.\frac{1}{5+x} - 5.\frac{1}{7-3x}(-3) - \frac{1}{3}.\frac{1}{8+5x}.5$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{3}{x} + \frac{1}{2(5+x)} + \frac{15}{7-3x} - \frac{5}{3(8+5x)}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3\sqrt{5+x}}{(7-3x)^5\sqrt[3]{8+5x}}\left[\frac{3}{x} + \frac{1}{2(5+x)} + \frac{15}{7-3x} - \frac{5}{3(8+5x)}\right]$$