Linear Inequalities

The mathematical expressions $(a > b)$ or $(a < b)$ are used to compare the two real numbers a and b i.e. it implies that 'a is greater than b' or 'a is smaller than b' respectively.

Suppose that if we say that salary of Mr. A is more than salary of Mr. B. Let Mr. B earns Rs. 20,000 per month, and the salary of Mr. A is x, then we can say that $x > 20000$. This is a linear inequation involving one variable.

Similarly, if we say that father's age is greater then 25 years more than the twice of the age of his child, then if father's age is x and child's age is y, then we can say that, $x > 2y + 25$. This is linear inequation in two variables.

GENERAL RULES

For any real numbers *a, b* and *c.*

(i) $a \geq b$ implies that a is greater than or equals to b. If any of the statements is correct, the statement $a \geq b$ holds true i.e. if we say $5 \geq 3$ or $5 \geq 5$. Then both of these statements are true as $5 > 3$ for the first case and 5 = 5 for the second case.

(ii) If $a > b \Rightarrow a + c > b + c$

(iii) If $a > b \Rightarrow a - c > b - c$

(iv) If $a > b \Rightarrow -a < -b$ i.e. $7 > 2 \Leftrightarrow -7 < -2$.

(v) If $a > b \Rightarrow ka > kb$ or $\frac{a}{k} > \frac{b}{k}$, where $k \in R^+$.

(vi) If $a > b \Rightarrow ka < kb$ or $\frac{a}{k} < \frac{b}{k}$, where $k \in R^-$

i.e. multiplying or dividing on both sides by a '–ve' quantity, reverses the sign of inequality. Say, if $x > 3$, then $1 > \frac{3}{x}$ hold true only if $x \in R^+$ but in the case $x \in R^-$, then $x > 3 \Rightarrow 1 < \frac{3}{x}$.

(vii) If $a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b}$.

1 INTERVALS

A subset of number line is called an interval if it contains all the real numbers lying between every pair of its elements.

Let $a, b \in R$, then the set

(i) $\{x : x \in R, a \leq x \leq b\}$ is called a closed interval and is denoted by [*a, b*].

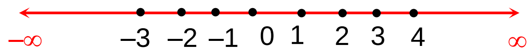
(ii) $\{x : x \in R, a < x < b\}$ is called an open interval and is denoted by (*a, b*), or]*a, b*[.

(iii) $\{x : x \in R, a \leq x < b\}$ is called left closed and right open interval and is denoted by [*a, b*) or [*a, b*[.

(iv) $\{x : x \in R, a < x \leq b\}$ is called left open and right closed interval and is denoted by (*a, b*] or]*a, b*].

(v) Discrete number are represented by curly { } brackets, e.g. {*a, b*}.

1.1 NUMBER LINE



The extension of number line is from $-\infty$ to $+\infty$ and any real number can be represented on this number line.

The solution of the inequations are shown on this number line as illustrated in the following examples

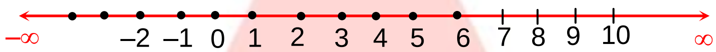
Illustration 1

Question: **Solve $x < 7$ for the following cases**

(a) $x \in I$ (b) $x \in N$ (c) $x \in R$

Solution: (a) $x \in I$

In this case $x = \{...... -2, -1, 0, 1, 2, 3, ...6\}$



The infinite solutions are represented by dots

(b) $x < 7,\ x \in N$

In this case $x = \{1, 2, 3, 4, 5, 6\}$

The finite solutions are represented by dots.

(c) $x < 7,\ x \in R$

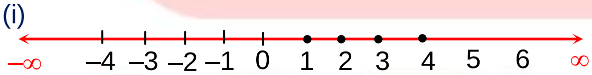
The infinite solutions are represented by a dark line as shown. A circle over 7 indicates that point 7 is not included in the solution $x \in (-\infty, 7)$.

Illustration 2

Question: **Show the solution of the linear inequation $-\frac{3}{2} \le x < 5$ on the number line.**

(i) when $x \in N$ (ii) when $x \in I$ (iii) when $x \in R$

Solution: (i)

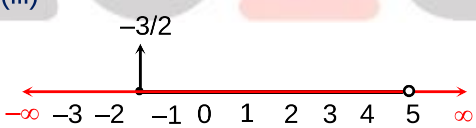


$x = \{1, 2, 3, 4\}$

(ii)

$x = \{-1, 0, 1, 2, 3, 4\}$

(iii)



$x \in \left[\frac{-3}{2}, 5\right)$

Important formulae/points

While representing the solution on the number line.

- If the end point is not included in the solution show it by hole i.e. o.
- If the end point is included in the solution show it by dot i.e. •.

General Rules:

- If $a > b \Rightarrow a + c > b + c$, where c is a real number.
- If $a > b \Rightarrow a - c > b - c$, where c is a real number.
- If $a > b \Rightarrow -a < -b$ i.e. $7 > 2 \Leftrightarrow -7 < -2$.
- If $a > b \Rightarrow ka > kb$ or $\frac{a}{k} > \frac{b}{k}$, where $k \in R^{+}$.
- If $a > b \Rightarrow ka < kb$ or $\frac{a}{k} < \frac{b}{k}$, where $k \in R^{-}$.
- If $a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b}$.

Intervals:

- $\{x : x \in R, a \le x \le b\}$ is called a closed interval and is denoted by [a, b].
- $\{x : x \in R, a < x < b\}$ is called an open interval and is denoted by (a, b), or]a, b[.

$\{x : x \in R, a \le x < b\}$ is called left closed and right open interval and is denoted by [a, b) or [a, b[.

$\{x : x \in R, a < x \le b\}$ is called left open and right closed interval and is denoted by (a, b] or]a, b].

Discrete number are represented by curly { } brackets, e.g. {a, b}.

2 SOLUTION OF LINEAR INEQUATION

The linear inequations are solved for set of real numbers. The method is to collect all terms involving the variable on one side of the inequation and the constant terms on the other side of the inequation, then use number line system to represent the solution. Consider the following examples, where in every case $x \in R$.

Illustration 3

Question: **Solve $2x - 3 > 4x + 5$.**

Solution: $2x - 3 > 4x + 5$

$\Rightarrow \quad (2x - 3) - 4x > (4x + 5) - 4x$ Rule-3

$\Rightarrow \quad -2x - 3 > 5$

or $\quad (-2x - 3) + 3 > 5 + 3$ Rule-2

$\Rightarrow \quad -2x > 8$

or $\quad \frac{-2x}{-2} < \frac{8}{-2} \quad \Rightarrow \quad x < -4$ Rule-6

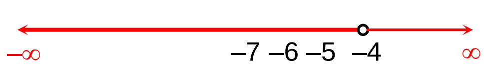


Illustration 4

Question: **Solve:** $x - 3 \le 5x + 2$.

Solution: Collecting terms involving variables on one side and terms involving constants on other side, we get

$-4x \le 5$ or $x \ge \frac{-5}{4}$ (dividing by –4)

Illustration 5

Question: **Solve:** $\frac{5-2x}{3} \le \frac{x}{6} - 5$.

Solution: Multiplying both sides by the LCM of 3 and 6, i.e. 6, we get

$2(5-2x) \le x - 30$

$\Rightarrow \quad 10 - 4x \le x - 30 \quad \Rightarrow \quad 5x \ge 40 \quad \Rightarrow \quad x \ge 8$

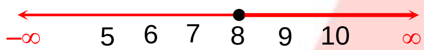


Illustration 6

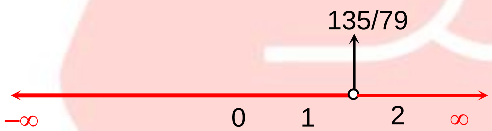
Question: **Solve:** $7 - \left(\frac{3x+5}{2}\right) > \frac{x}{3} + \frac{4x}{5}$.

Solution: Multiplying by 30 on both sides, we get

$210 - 15(3x+5) > 10x + 24x$

$\Rightarrow \quad 210 - 45x - 75 > 34x \quad \Rightarrow \quad 79x < 135$

or $\quad x < \frac{135}{79}$



Important formulae/points

- *To solve linear inequation, the method is to collect all terms involving the variable on one side of the inequation and the constant terms on the other side of the inequation, then use number line system to represent the solution.*

3 SOLUTION OF SYSTEM OF LINEAR INEQUATIONS IN ONE VARIABLE

System of linear inequations consists of more than one linear inequations. To solve a system, we first solve each linear inequation separately and then look for the common values of the variable satisfying each of the linear inequations.

Illustration 7

Question: **Solve the following system of inequations:**

$\mathbf{2x - 3 > x + 1,\ 3x + 5 \ge 2}$

Solution: $2x - 3 > x + 1$

$\Rightarrow \quad x > 4 \quad \ldots(i)$

Also, $3x + 5 \ge 2$

$\Rightarrow \quad 3x \ge -3$

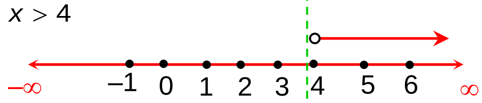
$\Rightarrow \quad x \ge -1 \quad \ldots(ii)$

The common solution of equations (i) and (ii) is $x > 4$ as required x is such that it is greater than or equal to –1 as well as greater than 4 $\Rightarrow \ x > 4$

Graphically:

$x \geq -1$

$x > 4$



$\Rightarrow x > 4$

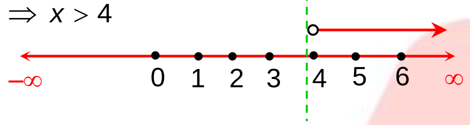
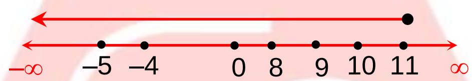


Illustration 8

Question: **Solve $5(2x-7)-3(2x+3)\leq 0$; $2x+19<6x+47$.**

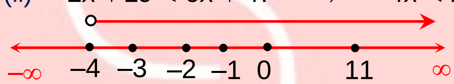
Solution: Solving inequation (i)

$10x-35-6x-9\leq 0$



$\Rightarrow \quad 4x-44\leq 0 \quad \Rightarrow \quad x\leq 11$

Solving inequation (ii) $2x+19<6x+47 \quad \Rightarrow \quad -4x<28 \quad \Rightarrow \quad x>-4$



The solution is the common values of x such $x\leq 11$ and $x>-4 \Rightarrow -4<x\leq 11$

The common part

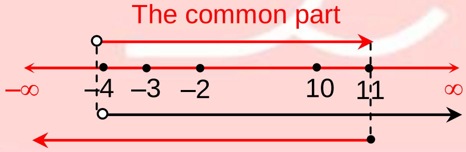
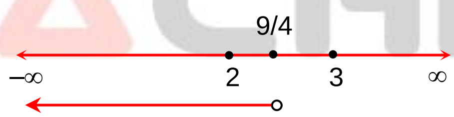


Illustration 9

Question: **Solve: $x+\frac{1}{2}>3x-4$ and $\frac{5x}{2}+\frac{3x-14}{3}\geq 2x$.**

Solution: Solving $x+\frac{1}{2}>3x-4$

$\Rightarrow \quad 2x+1>2(3x-4) \Rightarrow \quad 2x+1>6x-8 \quad \Rightarrow \quad 4x<9 \quad \Rightarrow \quad x<\frac{9}{4}$

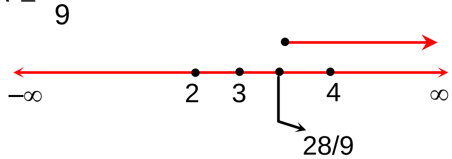


Solving $\frac{5x}{2}+\frac{3x-14}{3}\geq 2x$

Multiplying both sides by 6, we get

$15x+2(3x-14)\geq 12x \quad \Rightarrow \quad 15x+6x-28\geq 12x \quad \Rightarrow \quad 21x-28\geq 12x$

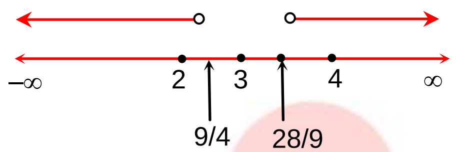
$\Rightarrow \quad 9x\geq 28 \quad \Rightarrow \quad x\geq \frac{28}{9}$



The solution is the common values of x, such that $x < \frac{9}{4}$ and $x \geq \frac{28}{9}$

We note that there is no value of x, which satisfies both inequation $\Rightarrow$ no solution

Graphically:



$\Rightarrow$ no common part

Illustration 10

Question: **Solve: $\frac{5}{4} < 2x - 3 \leq \frac{20}{3}$.**

Solution: Consider $\frac{5}{4} < 2x - 3$ and $2x - 3 \leq \frac{20}{3}$

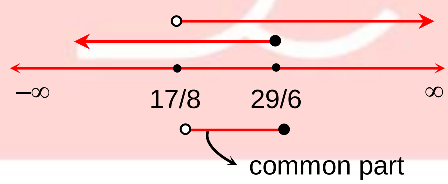
Separately, $\frac{5}{4} < 2x - 3 \Rightarrow 5 < 8x - 12 \Rightarrow 8x > 17 \Rightarrow x > \frac{17}{8}$

Similarly, $2x - 3 \leq \frac{20}{3} \Rightarrow 6x - 9 \leq 20 \Rightarrow 6x \leq 29 \Rightarrow x \leq \frac{29}{6}$

Common values of x are such that

$x > \frac{17}{8}$ and $x \leq \frac{29}{6} \Rightarrow \frac{17}{8} < x \leq \frac{29}{6}$

Graphically,



Alternate:

$\frac{5}{4} < 2x - 3 \leq \frac{20}{3}$

Multiplying the inequation by 12,

$15 < 12(2x - 3) \leq 80 \Rightarrow 15 < 24x - 36 \leq 80 \Rightarrow 15 + 36 < 24x - 36 + 36 \leq 80 + 36$

$\Rightarrow 51 < 24x \leq 116 \Rightarrow \frac{51}{24} < x \leq \frac{116}{24} \Rightarrow \frac{17}{8} < x \leq \frac{29}{6}$

Illustration 11

Question: **Solve $\frac{x-3}{x+4} > 0$, $\frac{2x-1}{x+1} < 1$.**

Solution: Taking $\frac{x-3}{x+4} > 0$

Multiplying both sides by $(x+4)^2$, we get $(x-3)(x+4) > 0$

Now, we have the product of $(x-3)$ and $(x+4)$.

It will be positive, if both of the values are positive or both of the values are negative.

$\Rightarrow x - 3 > 0$ and $x + 4 > 0 \Rightarrow x > 3$ and $x > -4 \Rightarrow x > 3$

OR $x - 3 < 0$ and $x + 4 < 0 \Rightarrow x < 3$ and $x < -4 \Rightarrow x < -4$

$\Rightarrow x > 3$ or $x < -4 \Rightarrow x \in (-\infty, -4) \cup (3, \infty)$

Taking, $\frac{2x-1}{x+1} < 1 \Rightarrow \frac{2x-1}{x+1} - 1 < 0 \Rightarrow \frac{2x-1-(x+1)}{x+1} < 0 \Rightarrow \frac{x-2}{x+1} < 0$

Multiplying both sides by $(x+1)^2$, we get $(x+1)(x-2)<0$

The product of $(x+1)$ and $(x-2)$ will be negative if one of them is positive and other is a negative quantity i.e. $x+1>0$ and $x-2<0$

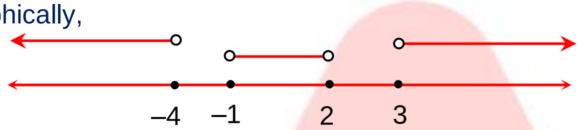
$\Rightarrow \quad x>-1$ and $x<2 \quad \Rightarrow \quad -1<x<2$ OR $x+1<0$ and $x-2>0$

$\Rightarrow \quad x<-1$ and $x>2 \quad \Rightarrow$ not possible.

Hence solution is $-1<x<2$ for the second equation.

We note that for $x \in (-\infty, -4) \cup (3, \infty)$ and $-1<x<2$, there is no common value of x and hence no solution.

Graphically,



Obviously, there is no common part, hence no solution.

Illustration 12

Question: **Solve the following system of inequations:**

$$\frac{x+3}{x-2} \le 2, \quad \frac{2x+5}{x+7} \ge 3.$$

Solution: The given inequations are

$$\frac{x+3}{x-2} \le 2 \quad \ldots(i) \quad \text{and} \quad \frac{2x+5}{x+7} \ge 3 \quad \ldots(ii)$$

(i) $\Rightarrow \quad \frac{x+3}{x-2} - 2 \le 0$

$\Rightarrow \quad \frac{x+3-2x+4}{x-2} \le 0$

$\Rightarrow \quad \frac{7-x}{x-2} \le 0$

$\therefore \quad 7-x \ge 0, \; x-2<0$

$\Rightarrow \quad x \le 7, \; x<2 \quad \Rightarrow \quad x<2 \quad \Rightarrow \quad x \in (-\infty, 2)$

or $\quad 7-x \le 0, \; x-2>0$

$\Rightarrow \quad x \ge 7, \; x>2 \quad \Rightarrow \quad x \ge 7 \quad \Rightarrow \quad x \in [7, \infty)$

$\therefore \quad x \in (-\infty, 2) \cup [7, \infty)$

(ii) $\Rightarrow \quad \frac{2x+5}{x+7} - 3 \ge 0$

$\Rightarrow \quad \frac{2x+5-3x-21}{x+7} \ge 0$

$\Rightarrow \quad \frac{-x-16}{x+7} \ge 0$

$\therefore \quad -x-16 \ge 0, \; x+7>0 \quad \Rightarrow \quad x \le -16, \; x>-7$

This is impossible

or $\quad -x-16 \le 0, \; x+7<0$

$\Rightarrow \quad x \ge -16, \; x<-7 \quad \Rightarrow \quad x \in [-16, -7)$

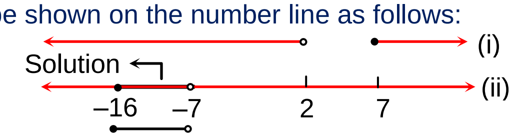
$\therefore \quad x \in [-16, -7)$

$\therefore$ Solution set of given system $= ((-\infty, 2) \cup [7, \infty)) \cap [-16, -7)$

$= [(-\infty, 2) \cap [-16, -7)] \cup [[7, \infty) \cap [-16, -7)]$

$= [-16, -7) \cup \phi = [-16, -7)$

The solution set can also be shown on the number line as follows:



Important formulae/points

- *To solve a system of linear inequations, we first solve each linear inequation separately and then look for the common values of the variable satisfying each of the linear inequations.*

In general, if $a, b \in R$ *and* $a < b$, *then*

- $(x-a)(x-b) > 0$ or $\frac{x-a}{x-b} > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)$
- $(x-a)(x-b) \geq 0 \Rightarrow x \in (-\infty, a] \cup [b, \infty)$
- $\frac{x-a}{x-b} \geq 0 \Rightarrow x \in (-\infty, a] \cup (b, \infty)$
- $(x-a)(x-b) < 0$ or $\frac{x-a}{x-b} < 0 \Rightarrow x \in (a, b)$
- $(x-a)(x-b) \leq 0 \Rightarrow x \in [a, b]$
- $\frac{x-a}{x-b} \leq 0 \Rightarrow x \in [a, b)$
- *Here, while solving* $\frac{ax+b}{cx+d} < k$, *we can't multiply both sides by (cx + d) in the starting as we have no knowledge about the value of (cx + d) whether it is positive or negative.*

4 GRAPHICAL SOLUTION OF LINEAR INEQUATION IN TWO VARIABLES

Any linear inequation in two variable is of the form $ax + by > c, ax + by \geq c, ax + by < c$ or $ax + by \leq c, (a \neq 0, b \neq 0)$. The graphical solution of the inequations will be discussed assuming *x, y* to be real numbers. Now consider the following examples.

Illustration 13

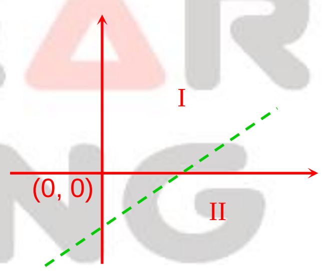
Question: **Solve $2x - 3y > 6$ graphically.**

Solution: First consider $2x - 3y = 6$.

This is equation of a straight line which divides the coordinate plane in two half planes.

Since inequation doesn't involve the sign of equality, so we will draw this line dotted or broken.

Now, consider any point which doesn't lie on this line, i.e. doesn't satisfy the equation $2x - 3y = 6$ for example take (0, 0) and check whether this point satisfies the inequation



$2x - 3y > 6$ or not.

$\Rightarrow \quad 2 \times 0 - 3 \times 0 > 6 \Rightarrow 0 > 6 \Rightarrow$ false statement.

Since (0, 0) is a point lying in the plane I and it doesn't satisfy the inequation, hence half plane II will be the required solution. This region is called solution region and it is shaded as shown:

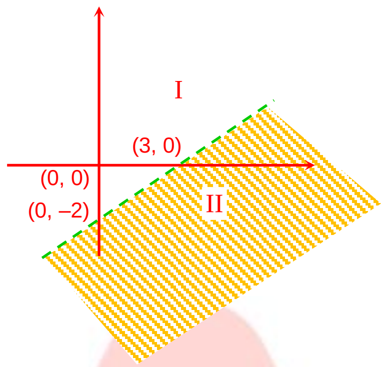


Illustration 14

Question: **Solve $x + 2y \le 4$ graphically.**

Solution: Since inequation involve equality, drawn the line $x + 2y = 4$ as shown (unbroken)

Putting $x = 0$, $y = 0$

$\Rightarrow \quad 0 + 2(0) \le 4 \;\Rightarrow\; 0 \le 4 \;\Rightarrow$ true statement

Hence region containing the origin is the solution region and hence shown shaded.

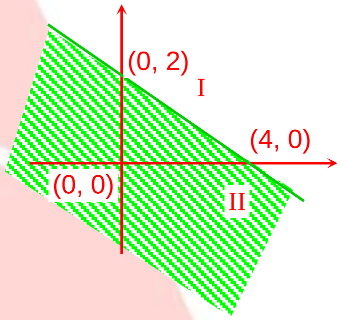


Illustration 15

Question: **Solve $3x + 2y \ge 0$. Show that no point in the third quadrant lies in the graph of inequation.**

Solution: Draw a continuous line corresponding to $3x + 2y = 0$.

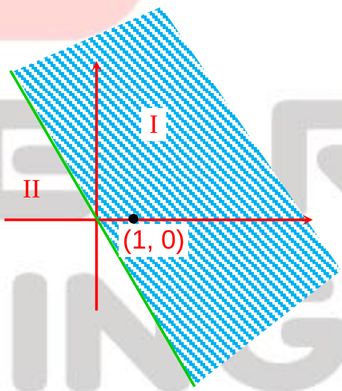
Put x= 1, y = 0 in the inequation

$\Rightarrow \quad 3(1) + 2(0) \ge 0$

$\Rightarrow \quad 3 \ge 0$

$\Rightarrow$ True statement.

Since the point (1, 0) lies in the I region, hence it is the solution region. Obviously no point in the 3rd quadrant lies in the solution.



Note: Here we can't take (0, 0) as it lies on the line $3x + 2y = 0$.

Illustration 16

Question: **Solve $x > -4$ graphically.**

Solution: Draw a dotted line corresponding to $x = -4$

Put $x = 0$ in the inequation $0 > -4 \;\Rightarrow$ true statement.

Hence region containing the origin is the solution region

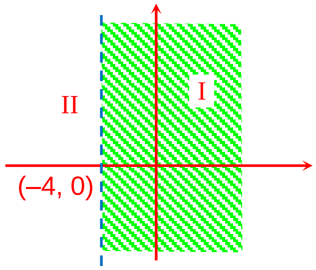


Illustration 17

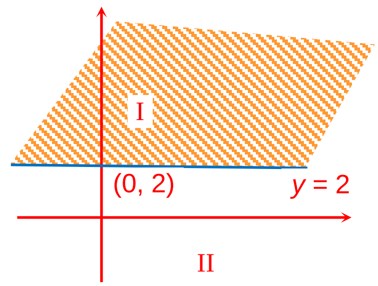
Question: **Solve $y \ge 2$ graphically.**

Solution: Draw a continuous line corresponding to $y = 2$

Putting $y = 0$ in the inequation $y \geq 2$

$\Rightarrow \quad 0 \geq 2 \Rightarrow$ False statement

Which implies that I is the solution region.



Important formulae/points

- *Any linear inequation in two variable is of the form* $ax+by>c, ax+by\geq c, ax+by<c$ *or* $ax+by\leq c, (a\neq 0, b\neq 0)$. *The graphical solution of the inequations is discussed assuming x, y to be real numbers.*
- *While giving the graphical solution for* $ax+by>c$, *line* $ax+by=c$ *which divides the plane should be dotted.*
- *While giving the graphical solution for* $ax+by\geq c$, *line* $ax+by=c$ *which divides the plane should be continuous.*
- *Any point which does not satisfy the equation of line* $ax+by=c$ *when put in the inequality, if it satisfies it, then shade the region containing the point, otherwise shade the other region. The shaded region is called solution region.*

5 SOLUTIONS OF SYSTEM OF LINEAR INEQUATIONS IN TWO VARIABLES

Illustration 18

Question: **Solve the following system of linear inequations graphically**
$\mathbf{x+y-2\geq 0}$, $\mathbf{3x-y\leq 0}$

Solution: Obtain the graphical solution for each of the inequations on the same set of axis and shade them. Thus obtained double shaded region is the required solution region.

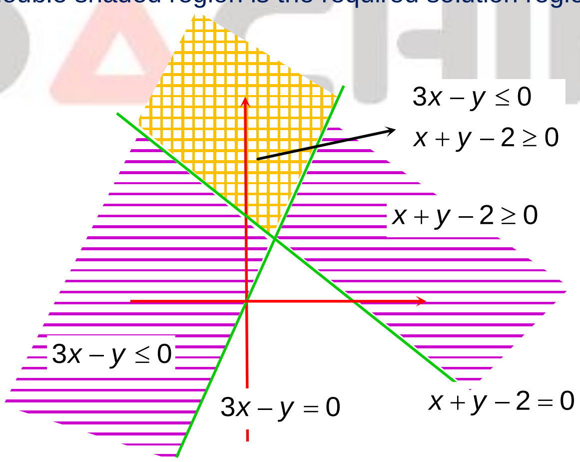
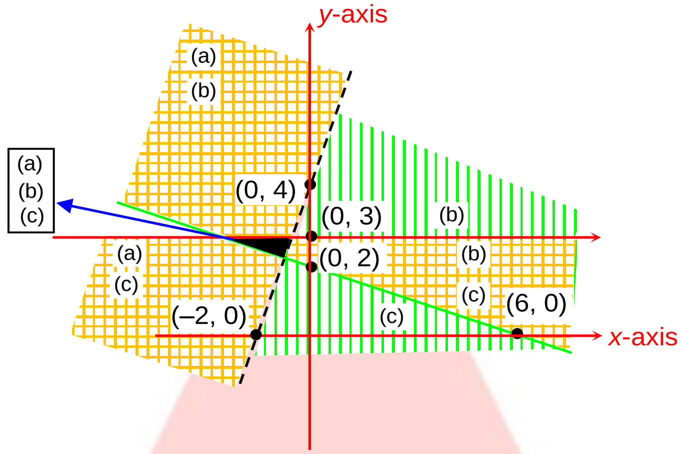


Illustration 19

Question: **Solve the following system of linear inequations graphically.**

(a) $y-2x-4>0$ (b) $x+3y-6\geq 0$ (c) $y\leq 3$

Solution: Obtain the graphical solution for each of the inequations on the same set of axis and shade them. Thus obtained black shaded region is the required solution region.



Note $x\geq 0$, $y\geq 0$ represent the first quadrant, so if present entire discussion will be limited only to first quadrant.

Illustration 20

Question: **Graphically solve the system of linear inequations:**

$$3x-4y+12\geq 0,\ \ 2x-y+2\geq 0$$

$$2x+3y-12\geq 0,\ \ x\leq 4,\ \ y\geq 2,\ \ x, y\geq 0$$

Solution: The linear inequations in the given system are

$3x-4y+12\geq 0$ …(i)

$2x-y+2\geq 0$ …(ii)

$2x+3y-12\geq 0$ …(iii)

$x\leq 4$ …(iv)

$y\geq 2$ …(v)

$x\geq 0$ …(vi)

$y\geq 0$ …(vii)

The line corresponding to (i) is $3x-4y+12=0$ …(viii)

(0, 3) and (–4, 0) are on (viii)

(0, 0) is not on (viii) and it lies in the half-plane of (i), if 3(0) – 4(0) + 12 ≥ 0, which is true.

∴ The closed half-plane containing the origin is the graph of (i)

The line corresponding to (ii) is $2x-y+2=0$ …(ix)

(0, 2) and (–1, 0) are on (ix)

(0, 0) is not on the line and it lies in the half-plane of (ii), if 2(0) – 0 + 2 ≥ 0, which is true.

∴ The closed half-plane containing the origin is the graph of (ii)

The line corresponding to (iii) is $2x+3y-12=0$ …(x)

(0, 4) and (6, 0) are on (x), (0, 0) is not on this line and it lies in the half-plane of (iii), if 2(0) + 3(0) – 12 ≥ 0, which is not true.

∴ The closed half-plane not containing the origin is the graph of (iii)

The inequation $x\leq 4$ represents the closed half-plane on the left of $x = 4$

The inequation $y\geq 2$ represents the closed half-plane above $y = 2$.

The inequation $x\geq 0$ and $y\geq 0$ represents the closed half-planes on the right of y-axis and above x-axis respectively.

The shaded region is the required graph of the given system of inequations.

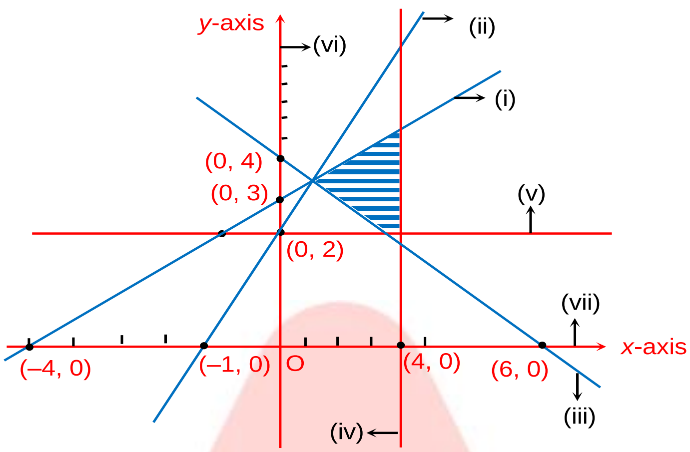


Illustration 21

Question: Solve $x - y \leq 3$, $3x + y \geq 6$, $x + 2y \leq 4$, $x \geq 0$, $y \geq 0$.

Solution:

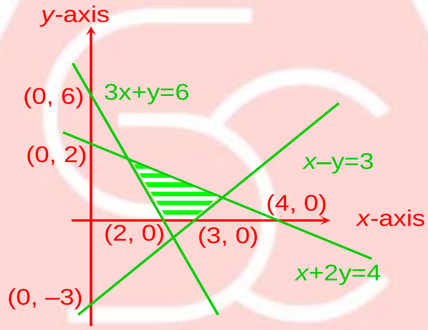
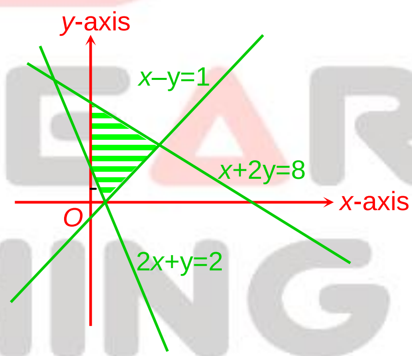


Illustration 22

Question:

Find the linear inequations for which the shaded area in the given figure is the solution set.



Solution:

The shaded area is bounded by the lines

$x + 2y = 8$ ...(i)

$x - y = 1$ ...(ii)

$2x + y = 2$ ...(iii)

$x = 0$ ...(iv)

Line (i) : The shaded area and the origin lies on the same side of $x + 2y = 8$.

$\therefore$ The corresponding inequation is $x + 2y \leq 8$ because $0 + 2(0) = 0 < 8$

Line (ii) : The shaded area and the origin lies on the same side of $x - y = 1$.

$\therefore$ The corresponding inequation is $x - y \leq 1$ because $0 – 0 = 0 < 1$

Line (iii) : The shaded area and the origin lies on the opposite sides of $2x + y = 2$.

$\therefore$ The corresponding inequation is $2x + y \geq 2$, because $2(0) + 0 = 0 < 2$

Line (iv) : The shaded area lies on the right of $x = 0$.

$\therefore$ The corresponding inequation is $x \geq 0$

The required system of inequations is $x + 2y \leq 8,\ x - y \leq 1,\ 2x + y \geq 2,\ x \geq 0$

Important formulae/points

- *Obtain the graphical solution for each of the inequations on the same set of axis and shade them. Thus obtained maximum shaded region is the required solution region.*
- $x \geq 0,\ y \geq 0$ *represent the first quadrant, so if present entire discussion will be limited only to first quadrant.*
- *In general, a system of two linear inequations in two variables can not be solved algebraically.*

6 INEQUALITIES RELATED TO MODULUS FUNCTION

In general $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Now consider the following inequalities and their solutions:

(i) $|x| \leq k,\ k > 0$

Solution: We have $|x| \leq k$ ...(i)

Case I: $x \geq 0$

$\therefore$ (i) $\Rightarrow$ $x \leq k$ $(\because\ |x| = x)$

$\therefore$ $0 \leq x \leq k$ i.e. $x \in [0, k]$

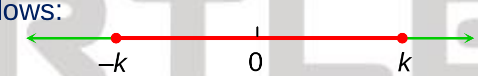
Case II: $x < 0$

$\therefore$ (i) $\Rightarrow$ $-x \leq k$ $(\because\ |x| = -x)$

$\Rightarrow$ $(-1)(-x) \geq (-1)k$ $\Rightarrow$ $x \geq -k$ $\Rightarrow$ $-k \leq x$

$\therefore$ $-k \leq x < 0$ i.e., $x \in [-k, 0)$

$\therefore$ The solution is $[-k, 0) \cup [0, k]$ which is same as $[-k, k]$. This solution can also be shown on the number line as follows:



Note: In particular, if $k = 0$, then the solution of the inequation $|x| \leq 0$ is {0}.

(ii) $|x| \geq k,\ k > 0$

Solution: We have $|x| \geq k$ ...(i)

Case-I : $x \geq 0$

(i) $\Rightarrow$ $x \geq k$ $(\because\ |x| = x)$

$\therefore$ $x \in [k, \infty)$

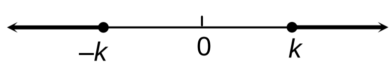
Case-II : $x < 0$

(i) $\Rightarrow$ $-x \geq k$ $(\because\ |x| = -x)$

$\Rightarrow$ $(-1)(-x) \leq (-1)k$ $\Rightarrow$ $x \leq -k$

$\therefore$ $x \in (-\infty, -k]$

$\therefore$ The solution is $(-\infty, -k] \cup [k, \infty)$. This solution can also be shown on the number line as follows:



Note: In particular, if $k = 0$, then the solution of the inequation $|x| \geq 0$ is $(-\infty, \infty)$.

Illustration 23

Question: **Solve the inequations: (i) $|7x-2|\le 11$ (ii) $|5-4x|>8$**

Solution: (i) $|7x-2|\le 11$ ...(i)

Case-I: $7x-2\ge 0$ i.e. $7x\ge 2$ or $x\ge \frac{2}{7}$

(i) $\Rightarrow$ $7x-2\le 11$ $(\because\ |7x-2|=7x-2)$

$\Rightarrow$ $7x\le 13$ $\Rightarrow$ $x\le \frac{13}{7}$ $\therefore$ $\frac{2}{7}\le x\le \frac{13}{7}$ i.e. $x\in\left[\frac{2}{7},\ \frac{13}{7}\right]$

Case-II: $7x-2<0$ i.e. $7x<2$ or $x<\frac{2}{7}$

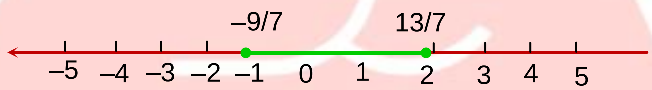
(i) $\Rightarrow$ $-(7x-2)\le 11$ $(\because\ |7x-2|=-(7x-2))$

$\Rightarrow$ $-7x+2\le 11$ $\Rightarrow$ $-7x\le 9$ $\Rightarrow$ $x\ge -\frac{9}{7}$ $\Rightarrow$ $-\frac{9}{7}\le x$

$\therefore$ $-\frac{9}{7}\le x<\frac{2}{7}$ i.e. $x\in\left[-\frac{9}{7},\ \frac{2}{7}\right)$

$\therefore$ The solution is $\left[-\frac{9}{7},\ \frac{2}{7}\right)\cup\left[\frac{2}{7},\ \frac{13}{7}\right]$, which is same as $\left[-\frac{9}{7},\ \frac{13}{7}\right]$.

This solution can also be shown on the number line as follows:



(ii) $|5-4x|>8$

Case-I: $5-4x\ge 0$ i.e. $5\ge 4x$ or $x\le \frac{5}{4}$

$\Rightarrow$ $5-4x>8$ $(\because\ |5-4x|=5-4x)$

$\Rightarrow$ $5-8>4x$ $\Rightarrow$ $4x<-3$ $\Rightarrow$ $x<-\frac{3}{4}$

$\therefore$ $x<-\frac{3}{4}$ i.e. $x\in\left(-\infty,\ -\frac{3}{4}\right)$ $\left(\because\ x<-\frac{3}{4}\ \Rightarrow\ x\le \frac{5}{4}\right)$

Case-II: $5-4x<0$ i.e. $5<4x$ or $x>\frac{5}{4}$

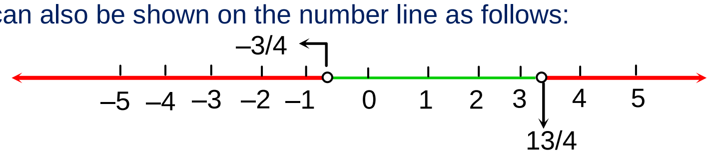
$\Rightarrow$ $-(5-4x)>8$ $(\because\ |5-4x|=-(5-4x))$

$\Rightarrow$ $-5+4x>8$ $\Rightarrow$ $4x>13$ $\Rightarrow$ $x>\frac{13}{4}$

$\therefore$ $x>\frac{13}{4}$ i.e. $x\in\left(\frac{13}{4},\ \infty\right)$ $\left(\because\ x>\frac{13}{4}\ \Rightarrow\ x>\frac{5}{4}\right)$

$\therefore$ The solution is $\left(-\infty,\ -\frac{3}{4}\right)\cup\left(\frac{13}{4},\ \infty\right)$.

This solution can also be shown on the number line as follows:



Important formulae/points

In general,

- *If* $|x| \geq k,\ k > 0 \Rightarrow\ x \in (-\infty, -k] \cup [k, \infty)$
- *If* $|x| > k,\ k > 0 \Rightarrow\ x \in (-\infty, -k) \cup (k, \infty)$
- *If* $|x| \leq k,\ k > 0 \Rightarrow\ x \in [-k, k]$
- *If* $|x| < k,\ k > 0 \Rightarrow\ x \in (-k,\ k)$

Solving $|x - x_0| \geq k, k > 0$

- *Under the case* $x - x_0 \geq 0$, *take* $|x - x_0| = x - x_0$ *and solve* $x - x_0 \geq k$
- *When* $x - x_0 < 0$, *take* $|x - x_0| = -(x - x_0)$ *and solve* $-(x - x_0) \geq k$
- *Combined the solutions obtained for the two cases.*

Illustration 24

Question: **The longest side of a triangle is 3 times the shortest side and the third side is 2 cm shorter than the longest side. If the perimeter of the triangle is at least 61 cm, find the minimum length of the shortest side.**

Solution: Let the sides of the triangle be a cm, b cm, c cm and $a > b > c$

$\therefore$ By the given conditions, $a = 3c$ …(i)

$b = a - 2$ …(ii)

and $a + b + c \geq 61$ …(iii)

(ii) $\Rightarrow$ $b = 3c - 2$ …(iv)

Putting the values of a and b from (i) and (iv) in (iii), we get

$$3c + (3c - 2) + c \geq 61 \ \Rightarrow\ 7c \geq 61 + 2 \ \Rightarrow\ c \geq \frac{63}{7} \ \Rightarrow\ c \geq 9$$

$\therefore$ Minimum length of the shortest side = 9 cm.

Illustration 25

Question: **A plumber can be paid under two schemes as given below:**

I : Rs. 600 and Rs. 50 per hour

II : Rs. 170 per hour

If the job takes *n* hours, for what values of *n* does the scheme I give the plumber the better wages?

Solution: Number of hours $= n$

Under scheme I, wage of plumber = Rs. (600 + 50 n)

Under scheme II, wage of plumber = Rs. 170 n

Let 600 + 50n > 170 n $\Rightarrow$ $600 > 120n \ \Rightarrow\ n < \frac{600}{120} = 5$

$\therefore$ Number of hours should be less than 5 hours.

Illustration 26

Question: **In drilling world's deepest hole, it was found that the temperature *T* in degree Celsius, *x* km below the surface of earth, was given by $T = 30 + 25(x - 3),\ 3 < x < 15$. At what depth, will the temperature be between 200°C and 300°C?**

Solution: We have, $200 < T < 300$ …(i)

Putting $T = 30 + 25(x - 3)$, (i) becomes $200 < 30 + 25(x - 3) < 300$

$\Rightarrow\ 200 - 30 < 30 + 25(x - 3) - 30 < 300 - 30$

$\Rightarrow\ 170 < 25(x - 3) < 270 \ \Rightarrow\ \frac{170}{25} < x - 3 < \frac{270}{25}$

$\Rightarrow\ 6.8 < x - 3 < 10.8 \ \Rightarrow\ 6.8 + 3 < (x - 3) + 3 < 10.8 + 3 \ \Rightarrow\ 9.8 < x < 13.8$

$\therefore$ Depth of the hole is between 9.8 km and 13.8 km.

Illustration 27

Question: **The water acidity in a pool is considered normal when the average pH reading of three daily measurements is between 7.2 and 7.8. If the first two pH readings are 7.48 and 7.85, find the range of pH value for the third reading that will result in the acidity level being normal.**

Solution: Let the third reading be x

$\therefore$ Average = $\dfrac{7.48+7.85+x}{3}=\dfrac{15.33+x}{3}$

By the given condition , average lies between 7.2 and 7.8

$\Rightarrow \quad 7.2<\dfrac{15.33+x}{3}<7.8 \quad \Rightarrow \quad 21.6<15.33+x<23.4 \quad \Rightarrow \quad 21.6-15.33<x<23.4-15.33$

$\Rightarrow \quad 6.27<x<8.07$

$\therefore$ Range of third reading is between 6.27 and 8.07.

Illustration 28

Question: **A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If we have 640 litres of the 8% solution,, how many litres of the 2% solution will have to be added?**

Solution: Let required quantity of 2% boric acid = x litres

$\therefore$ Quantity of mixture = $(640 + x)$ litres

By the given conditions,

8% of 640 + 2% of x > 4% of $(640 + x)$ …(i)

and 8% of 640 + 2% of x < 6% of $(640 + x)$ …(ii)

(i) $\Rightarrow \left(\dfrac{8}{100}\right)640+\left(\dfrac{2}{100}\right)x>\left(\dfrac{4}{100}\right)(640+x)$

$\Rightarrow \quad 5120+2x>2560+4x \quad \Rightarrow \quad 2560>2x \quad \Rightarrow \quad x<1280$ …(iii)

(ii) $\Rightarrow \left(\dfrac{8}{100}\right)640+\left(\dfrac{2}{100}\right)x<\left(\dfrac{6}{100}\right)(640+x)$

$\Rightarrow \quad 5120+2x<3840+6x \quad \Rightarrow \quad 1280<4x \quad \Rightarrow \quad x>320$ ..(iv)

(iii) and (iv) $\Rightarrow \quad 320<x<1280$

$\therefore$ Quantity of 2% boric acid lies between 320 litres and 1280 litres.