LINEAR PROGRAMMING

1 INTRODUCTION

The term programming means planning and it refers to a particular plan of action from amongst several alternatives for maximizing profit or minimizing cost etc. Programming problems deal with determining optimal allocation of limited recourses to meet the given objectives, such as least cost maximum profit, highest margin or least time, when resources have alternative uses.

The term **'Linear'** means that all inequations or equations used and the function to be maximized or minimized are linear. That is why linear programming deals with that class of problems for which all relations among the variables involved are linear.

Formally, linear programming deals with the optimization (maximization or minimization) of a linear function of a number of variables subject to a number of conditions on the variables, in the form of linear inequations or equations in variables involved.

In this chapter, we shall discuss mathematical formulation of linear programming problems that arise in trade, industry, commerce and military operations. We shall also discuss some elementary techniques to solve linear programming problems in two variables only.

2 LINEAR PROGRAMMING PROBLEMS

In this section, we shall discuss the general form of a linear programming problem. To give the general description of a linear programming problem, let us consider the following problem:

Suppose that a furniture dealer makes two products viz. chairs and tables. Processing of these products is done on two machines *A* and *B*. A chair requires 2 hours on machine *A* and 6 hours on machine *B*. A table requires 4 hours on machine *A* and 2 hours on machine *B*. There are 16 hours of time per day available on machine *A* and 20 hours on machine *B*. Profits gained by the manufacturer from a chair and a table are Rs 30 and Rs 50 respectively. The manufacturer is willing to know the daily product of each of the two products to maximize his profit.

The above data can be put in the following tabular form:

Item	Chair	Table	Maximum available time
Machine *A*	2 hrs	4 hrs	16 hrs
Machine *B*	6 hrs	2 hrs	20 hrs
Profit (in Rs)	Rs 30	Rs 50	

To maximize his profit, suppose that the manufacturer produces *x* chairs and *y* tables per day. It is given that a chair requires 2 hours on machine *A* and a table requires 4 hours on machine *A*. Hence, the total time taken by machine *A* to produce *x* chairs and *y* tables is $2x + 4y$. This must be less than or equal to the total hours available on machine *A*. Hence $2x + 4y \le 16$. Similarly, for machine *B*, we have

$$6x + 2y \le 20.$$

The total profit for *x* chairs and *y* tables is $30x + 50y$. Since the number of chairs and tables is never negative. Therefore, $x \ge 0$ and $y \ge 0$.

Thus, we have to maximize

Profit = $30x + 50y$

Subject to the constraints

$$2x + 4y \le 16$$

$$6x + 2y \le 20$$

and $x \ge 0, y \ge 0$

Out of all the points (x, y) in the solution set of the above linear constraints, the manufacturer has to choose that point, or those points for which the profit $30x + 50y$ has the maximum value.

In the above discussion if a chair costs Rs 250 and a table costs Rs 300 then the total cost of producing x chair and y tables is $250x + 300y$. Now, the manufacturer will be interested to choose that point, or those points, in the solution set of the above linear constraints for which the cost $250x + 300y$ has the minimum value.

The two situations discussed above given the description of a type of linear programming problems. In the above discussion, the profit function = $30x + 50y$ or the cost function $= 250x + 300y$ is known as the objective function. The inequations $2x + 4y \leq 16$, $6x + 2y \leq 20$ are known as the constraints and $x \geq 0$, $y \geq 0$ are known as the non-negativity restrictions.

The general mathematical description of a linear programming problem (LPP) is given below:

Optimize $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$

Subject to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n (\leq, =, \geq) b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n (\leq, =, \geq) b_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n (\leq, =, \geq) b_m$$
$$x_1, x_2, x_3, ..., x_n \geq 0$$

where all a_{ij}'s, b_i's and c_j's are constants and x_j's are variables.

3 CONSTRUCTION OF LINEAR PROGRAMMING MODEL

This section illustrates the basic elements of a linear programming model by using a simple two variable example. The results provide concrete ideas for the solution and interpretation of the general linear programming problems.

Example: Reddy Mikks produces both interior and exterior paints from two raw materials. M1 and M2. The following table provides the basic data of the problem:

	Tons of raw material per ton of		
	Exterior paint	Interior paint	Maximum daily availability (tons)
Raw material, M1	6	4	24
Raw material, M2	1	2	6
Profit per ton ($1000)	5	4	

A market survey restricts the maximum daily demand of interior paint to 2 tons. Additionally, the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. Ready Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximize the total daily profit.

The LP model, includes three basic elements.

1. Decision variables that we seek to determine
2. Objective (goal) that we aim to optimize
3. Constraints that we need to satisfy

- The proper definition of the decision variables is an essential first step toward the development of the model. Once the variables are defined, the task of constructing the objective function and the constrains should not be too difficult.
  For the Reddy Mikks problem, we need to determine the amounts to be produced of exterior and interior paint. The variables of the model are thus defined as x = Tons produced daily of exterior paint, y = Tons produced daily of interior paint
- Using these definitions, the next task is to construct the objective function. A logical objective for the company is to increase as much as possible (i.e., maximize) the total daily profit from both exterior and interior paints. Letting z represent the total daily profit (in thousands of dollars), we get

$$z = 5x + 4y$$

The objective of the company is

Maximize $z = 5x + 4y$

- The last element of the model deals with the constraints that restrict raw materials usage and demand. The raw materials restrictions are expressed verbally as

(usage of a raw material by both paints) ≤ (maximum raw material availability)

From the data of the problem that

Usage of raw material $M1 = 6x + 4y$ tons

Usage of raw material $M2 = 1x + 2y$ tons

Because the daily availabilities of raw materials $M1$ and $M2$ are limited to 24 and 6 tons respectively, the associated restrictions are given as

$6x + 4y \leq 24$ (Raw material $M1$)

$x + 2y \leq 6$ (Raw material $M2$)

There are two types of demand restrictions (1). Maximum daily demand of interior paint is limited to 2 tons, and (2) excess of daily production of interior paint over that of exterior paint is at most 1 ton. The first restriction is straightforward and is expressed as $y \leq 2$. The second restriction can be translated to state that the difference between the daily production of interior and exterior paints, $y - x$, does not exceed 1 ton –that is, $y - x \leq 1$.

An implicit (or "understood-to-be") restriction on the model is that the variables x and y must not be negative. We thus add the non-negativity restrictions, $x \geq 0$, $y \geq 0$ to account for this requirement.

The complete Reddy Mikks model is written as

Maximize $z = 5x + 4y$

Subject to

$6x + 4y \leq 24$

$x + 2y \leq 6$

$-x + y \leq 1$

$y \leq 2$

and $x, y \geq 0$

Any solution that satisfies all the constraints of the model is a **feasible solution**. For example, the solution x = 3 tons and y = 1 ton is feasible because it does not violate any of the constraints, including the non-negativity restrictions. To verify this result, we substitute (x = 3, y = 1) in the left-hand side of each constraint to make sure that the inequalities are satisfied.

For example, in the first constraint, $6x + 4y = 6 \times 3 + 4 \times 1 = 22$, which is less than the right-hand side of the constraint (=24). The value of the objective function associated with the solution (x = 3, y = 1) is $z = 5 \times 3 + 4 \times 1 = 19$ (thousand dollars).

From the standpoint of the entire model, what we really are interested in is the **optimum feasible solution** that yields the maximum total profit. With some reflection, you should conclude that the model has a large (in fact, infinite) number of feasible solutions. As such, it is impossible to use direct substitutions to determine the optimum. Instead, we need efficient procedures that will locate the optimum solution systematically. The graphical method in next section is one of the procedure for finding optimal solution.

Illustration 1

Question: **A manufacturer of a line of patent medicines is preparing a production plan on medicines *A* and *B*. There are sufficient ingredients available to make 20, 000 bottles of *A* and 40,000 bottles of *B* but there are only 45,000 bottles into which either of the medicines can be put. Further more, it takes 3 hours to prepare enough material to fill 1000 bottles of *A*, it takes one hour to prepare enough material to fill 1000 bottles of *B* and there are 66 hours available for this operation. The profit is Rs 8 per bottle for *A* and Rs 7 per bottle for *B*. Formulate this problem as a linear programming problem.**

Solution: Suppose the manufacturer produces x bottles of medicines A and y bottles of medicine B.
Since the profit is Rs 8 per bottle for A and Rs 7 per bottle for B.
So, total profit in producing x bottles of medicine A and y bottles of medicine B is
Rs $(8x + 7y)$
Let Z denote the total profit. Then, $Z = 8x + 7y$
Since 1000 bottles of medicine A are prepared in 3 hours.
So, time required to prepare x bottles of medicine $A = \frac{3x}{1000}$ hours.
It is given that 1000 bottles of medicine B are prepared in 1 hour.
$\therefore$ Time required to prepare y bottles of medicine $B = \frac{y}{1000}$ hours.
Thus, total time required to prepare x bottles of medicine A and y bottles of medicine B is $\frac{3x}{1000} + \frac{y}{1000}$ hours.
But, the total time available for this operation is 66 hours.

$$\therefore \quad \frac{3x}{1000} + \frac{y}{1000} \le 66 \quad \Rightarrow \quad 3x + y \le 66000$$

Since there are only 45,000 bottles into which the medicines can be put.

$$\therefore \quad x + y \le 45{,}000$$

It is given that the ingredients are available for 20,000 bottles of A and 40,000 bottles of B.

$$\therefore \quad x \le 20{,}000 \text{ and } y \le 40{,}000$$

Since the number of bottles can not be negative. Therefore, $x \ge 0,\ y \ge 0$
Hence, the mathematical formulation of the given LPP is as follows:
Maximize $Z = 8x + 7y$
Subject to $3x + y \le 66{,}000;\ \ x + y \le 45{,}000;\ \ x \le 20{,}000;\ \ y \le 40{,}000$ and $x \ge 0,\ y \ge 0$

4 GRAPHICAL LINEAR PROGRAMMING SOLUTIONS

This section shows how a two variable linear programming model is solved graphically. Although two variable models rarely occur in practice (where a typical linear programming model may include thousands of variables and constraints), the ideas gleaned from the graphical procedure lay the foundation for the development of the general solution technique.

The graphical procedure includes two basic steps:

- The determination of the solution space that defines the feasible solutions that satisfy all the constraints of the model.
- The determination of the optimum solution from among all the points in the feasible solution space.

The procedure is described for both a maximization and a minimization objective function.

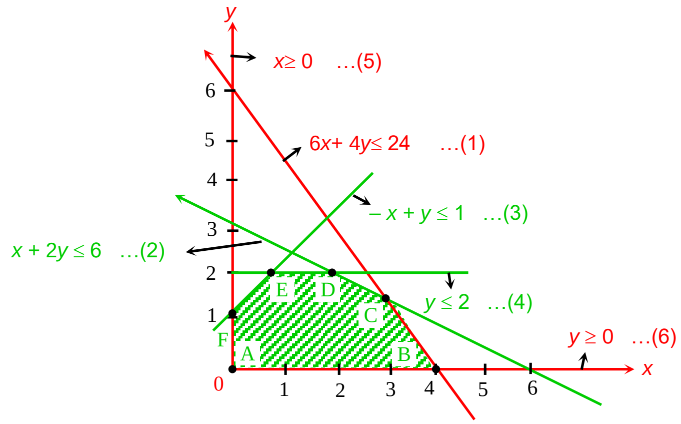
4.1 Solution of a Maximization Model

Example1:

We use the Reddy Mikks model to illustrate the two steps of the graphical procedure.

Step1: Determination of the Feasible Solution Space:

First, as shown in figure, let the horizontal axis x and the vertical axis y represent the exterior paint, interior paint, variables respectively. Next, consider the non negativity restrictions $x \ge 0$ and $y \ge 0$. These two constraints restrict the solution space area to the first quadrant (which lies above the x-axis and to the right of the y-axis).



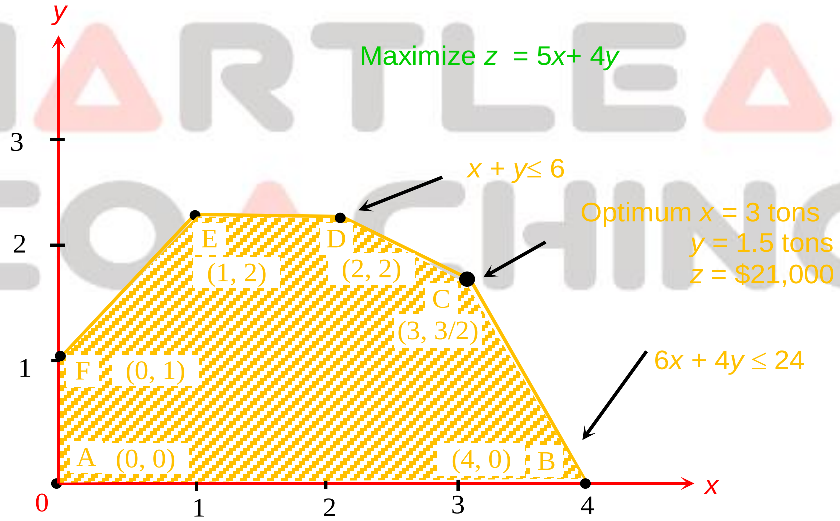
The easiest way of accounting for the remaining four constraints is to replace the inequalities with equations and then plot the resulting straight lines. For example, the inequality $6x+4y\le 24$ is replaced with the straight line $6x+4y=24$. To plot this line, we need two distinct point, which can be secured by first setting $x=0$ to obtain $y = 6$ and then setting $y = 0$ to obtain $x = 4$. Thus, the line passes through the two points (0, 6) and (4, 0) as shown by line (1) in figure.

Next, we consider the effect of the inequality. All the inequality does is to divide the (x, y) plane into two (half) spaces that occur on both sides of the plotted line; one side satisfies the inequality, and the other one does not. A procedure for determining the feasible side is to use the origin (0, 0) as a reference point. For example, for the first constraint, (0, 0) satisfies $6x+4y\le 24$ (that is, 6 × 0 + 4 × 0 = 0, which is less than 24). This means that the feasible side of the constraints $6x+4y\le 24$ includes the origin. This result is shown by the directional arrow associated with constraint (1) in figure.

In general, if the origin does not satisfy the inequality, then the directional arrow must point in the opposite side of (0, 0). Also, if the line happens to pass through the origin, then we can choose another reference point to effect the desired result.

Step2: Determination of the Optimum Solution:

Figure below, provides the feasible solution space that is satisfied by all the constraints of the model. This space is delineated by the line segments joining the corner points *A, B, C, D, E* and *F*. Any point within or on the boundary of the space *ABCDEF* is a feasible point, in the sense that it satisfies all the constraints. Because the feasible space *ABCDEF* consists of an infinite number of points.

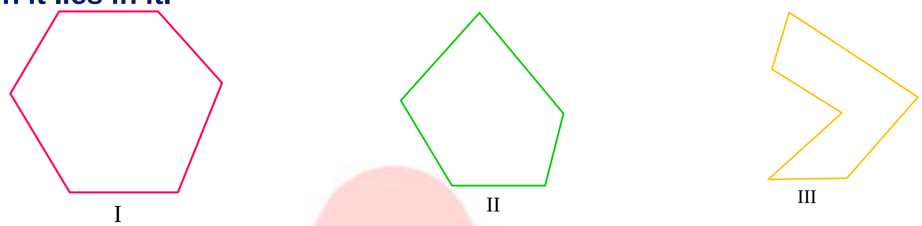


It is not evident how we should go about finding a point that gives a maximum value of the objective function $Z=5x+4y$. To handle this situation, we use the following theorems which are fundamental in solving linear programming problems.

Theorem1: Let *R* be the feasible region **(convex set)*** for a linear programming problem and let $Z=ax+by$ be the objective function. When *Z* has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a **corner point**** (vertex) of the feasible region.

Theorem2: Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is **bounded *****, then the objective function Z has both a maximum and a minimum value of R and each of these occurs at a corner point (vertex) of R.

* **A set is a convex set, if every point on the line segment joining any two points in it lies in it.**



In figures (I) and (II) the polygons are convex sets whereas polygon in figure (III) is not a convex set.

** *A corner point of a feasible region is a point in the region which is the intersection of two boundary lines.*

*** *A feasible region of a system of linear inequalities is said to be bounded if it can be enclosed within a circle. Otherwise, it is called unbounded. Unbounded means that the feasible region does extend indefinitely in any direction.*

Remark: If R is unbounded, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R (by theorem 1).

In the above example, the corner points (vertices) of the bounded (feasible) region are *A, B, C, D, E* and *F* and it is easy to find their coordinate as (0, 0), (4, 0), (3, 3/2), (2, 2), (1, 2) and (0, 1) respectively. Let us now compute the value of Z at these points.

We have

Vertex of the feasible region	Corresponding value of Z (in Rs)
$A(0, 0)$	0
$B(4, 0)$	20
$C(3, 3/2)$	21
$D(2, 2)$	18
$E(1, 2)$	13
$F(0, 1)$	4

We observe that the optimum solution is given by point C.

The solution yields $x = 3$ and $y = 1.5$ with $z = 5 \times 3 + 4 \times 1.5 = 21$. This means that the optimum daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint will yield a daily profit of $21,000.

This method of solving linear programming problem is referred as **Corner Point Method**. The method comprises of the following steps:

1. Find the feasible region of the linear programming problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines interesting at that point.
2. Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m, respectively denote the largest and smallest values of these points.
3. (i) When the feasible region is bounded, M and m are the maximum and minimum values of Z.
   (ii) In case, the feasible region is unbounded, we have:
   (a) M is the maximum value of Z, if the open half plane determined by $ax + by > M$

has no point in common with the feasible region. Otherwise, Z has no maximum value.

(b) Similarly, m is the minimum value of Z, if the open half plane determined by $ax+by<m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

Remark:

If two corner points of the feasible region are both optimal solutions of the same type, i.e., both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.

We will now illustrate these steps of **corner point method** by considering some examples.

Illustration 2

Question: **Solve the following problem graphically:**

Minimize and maximize $Z=3x+9y$ ...(i)

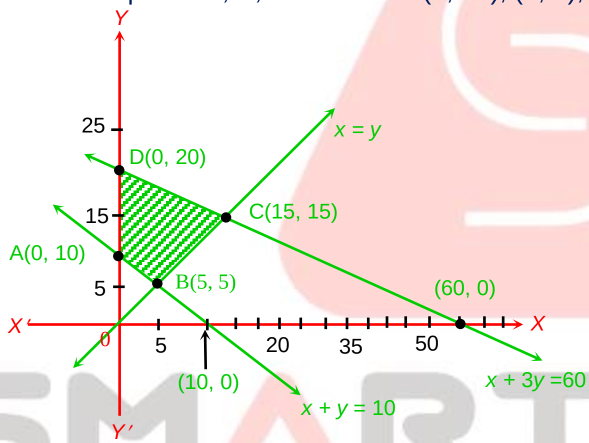
Subject to the constraints $x+3y\le 60$ ...(ii)

$x+y\ge 10$ ...(iii)

$x\le y$ ...(iv)

$x\ge 0,\ y\ge 0$ ...(v)

Solution: First of all, let us graph the feasible region of the system of linear inequalities (ii) to (v). The feasible region *ABCD* is shown in the figure. Note that the region is bounded. The coordinates of the corner points *A, B, C* and *D* are (0, 10), (5, 5), (15, 15) and (0, 20) respectively.



Corner Point	Corresponding values of Z = 3x + 9y
A(0, 0)	90
B(5, 5)	**60** ←
C(15, 15)	**180** ←
D(0, 20)	**180**

We now find the minimum and maximum value of Z. From the table, we find that the minimum value of Z is 60 at the point B(5, 5) of the feasible region.

The maximum value of Z on the feasible region occurs at the two corner points C (15, 15) and D(0, 20) and it is 180 in each case.

Remark:

Observe that in the above example, the problem has multiple optimal solutions at the corner points C and D i.e., the both points produce same a maximum value 180. In such case, you can see that every point on the line segment CD joining the two corner points C and D also give the same maximum value. Same is also true in the case if the two points produce same minimum value. For example mid-point of line joining points C and D is P(15/2, 35/2) which lies on CD. Now Z at P(15/2, 35/2) is 180, which is same as Z at C and D. Similarly, every point on line CD gives the same value of Z.

5 DIFFERENT TYPES OF LINEAR PROGRAMMING PROBLEMS

A few important linear programming problems are listed below:

1. Manufacturing problems

In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hour per unit of product, warehouse space per unit of the output etc., in order to make maximum profit.

2. Diet problems

In these problems, we determine the amount of different kinds of constituents/ nutrients which should be included in a diet so as to minimize the cost of the desired diet such that it contains a certain minimum amount of each constituent/ nutrients.

3. Transportation problems

In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants/ factories situated at different locations to different markets.

Illustration 3

Question: **A cooperative society of farmers has 50 hectare of land to grow two crops *X* and *Y*. The profit from crops *X* and *Y* per hectare are estimated as Rs. 10,500 and Rs. 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops *X* and *Y* at rates of 20 litres and 10 litres per hectare. Further, no more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so as to maximize the total profit of the society?**

Solution: Let x hectare of land be allocated to crop X and y hectare to crop Y.

Obviously, $x \geq 0$, $y \geq 0$

Profit per hectare on crop X = Rs. 10500

Profit per hectare on crop Y = Rs. 9000

Therefore, total profit = Rs. $(10500x + 9000y)$

The mathematical formulation of the problem is as follows:

Maximize $Z = 10500x + 9000y$

Subject to the constraints:

$x + y \leq 50$ (constraint related to land) …(i)

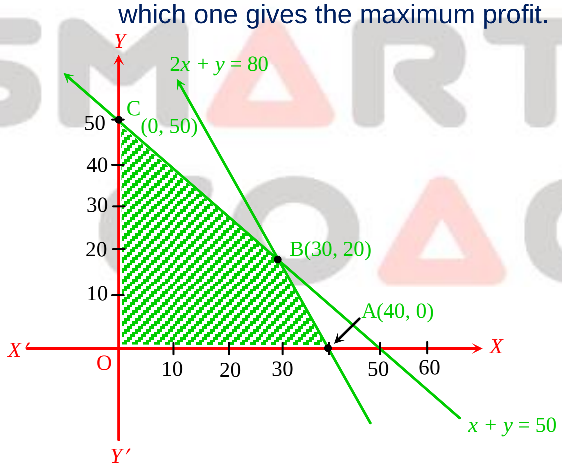
$20x + 10y \leq 800$ (constraint related to use of herbicide),

i.e., $2x + y \leq 80$ …(ii)

$x \geq 0$, $y \geq 0$ (non negative constraint) …(iii)

Let us draw the graph of the system of inequalities (i) to (iii). The feasible region *OABC* is shown (shaded) in the figure. Observe that the feasible region is bounded.

The coordinates of the corner points *O, A, B* and *C* are (0, 0), (40, 0), (30, 20) and (0, 50) respectively. Let us evaluate the objective function $Z = 10500x + 9000y$ at these vertices to find which one gives the maximum profit.



Corner Point	$Z = 10500x + 9000y$
O(0, 0)	0
A(40, 0)	420000
B(30, 20)	**495000** ← maximum
C(0, 50)	450000

Hence, the society will get the maximum profit of Rs. 4,95,000 by allocating 30 hectares for crop *X* and 20 hectares for crop *Y*.

Illustration 4

Question: **A house wife wishes to mix together two kinds of food *X* and *Y*, in such a way that the mixture contains at least 10 units of vitamin *A*, 12 units of vitamin *B* and 8 units of vitamin *C*.**

The vitamin contents of one kg of food is given below:

	Vitamin A	Vitamin B	Vitamin C
Food *X*:	1	2	3
Food *Y*:	2	2	1

One kg of food *X* costs Rs 6 and one kg of food *Y* costs Rs 10. Find the least cost of the mixture which will produce the diet.

Solution: Let x kg of food X and y kg of food Y are mixed together to make the mixture.
Since one kg of food X contains one unit of vitamin A and one kg of food Y contains 2 units of vitamin A. Therefore, x kg of food X and y kg of food Y will contain $x + 2y$ units of vitamin A.
But the mixture should contain at least 10 units of vitamin A. Therefore,

$$x + 2y \geq 10$$

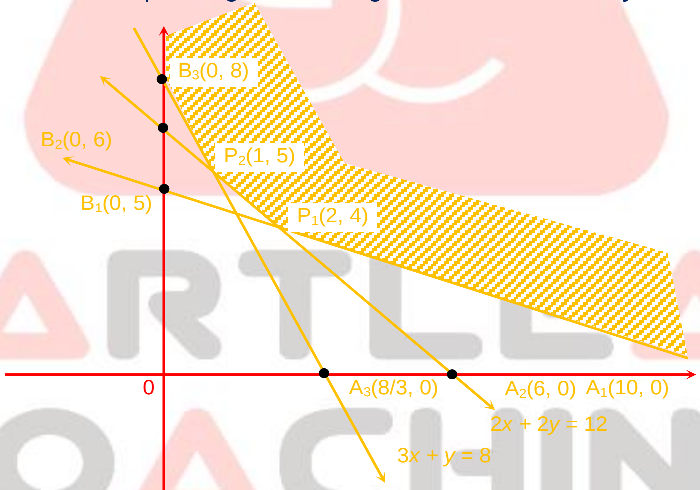
Similarly, x kg of food X and y kg of food Y will produce $2x + 2y$ units of vitamin B and $3x + y$ units of vitamin C.
But the minimum requirements of vitamins B and C are respectively of 12 and 8 units.

$\therefore \quad 2x + 2y \geq 12$ and $3x + y \geq 8$

Since the quantity of food X and food Y cannot be negative.

$\therefore \quad x \geq 0,\ y \geq 0$

It is given that one kg of food X costs Rs 6 and one kg of food Y costs Rs 10.
So, x kg of food X and y kg of food Y will cost Rs $(6x + 10y)$.
Thus, the given linear programming problem is
Minimize $Z = 6x + 10y$
Subject to $x + 2y \geq 10;\ 2x + 2y \geq 12;\ 3x + y \geq 8$ and $x \geq 0,\ y \geq 0$
To solve this LPP, we draw the lines $x + 2y = 10,\ 2x + 2y = 12$ and $3x + y = 8$
The feasible region of the LPP is shaded in figure:
The coordinates of the vertices (corner points) of shaded feasible region $A_1 P_1 P_2 B_3$ are $A_1(10, 0), P_1(2, 4), P_2(1, 5)$ and $B_3(0, 8)$. These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously.



The values of the objective function at these points are given in the following table:

Point (x, y)	Value of the objective function $Z = 6x + 10y$
$A_1(10, 0)$	$Z = 6 \times 10 + 10 \times 0 = 60$
$A_2(2, 4)$	$Z = 6 \times 2 + 10 \times 4 = 52$
$P_2(1, 5)$	$Z = 6 \times 1 + 10 \times 5 = 56$
$B_3(0, 8)$	$Z = 6 \times 0 + 10 \times 8 = 80$

Clearly, Z is minimum at $x = 2$ and $y = 4$. The minimum value of Z is 52.
Hence, the least cost of the mixture is Rs 52.

Illustration 5

Question: **An oil company requires 12,000, 20,000 and 15,000 barrels of high-grade, medium grade and**

low grade oil, respectively. Refinery *A* produces 100, 300 and 200 barrels per day of high-grade, medium-grade and low-grade oil, respectively, while refinery *B* produces 200, 400 and 100 barrels per day of high-grade, medium-grade and low-grade oil, respectively. If refinery *A* costs Rs 400 per day and refinery *B* costs Rs 300 per day to operate, how many days should each be run to minimize costs while satisfying requirements.

Solution: The given data may be put in the following tabular form:

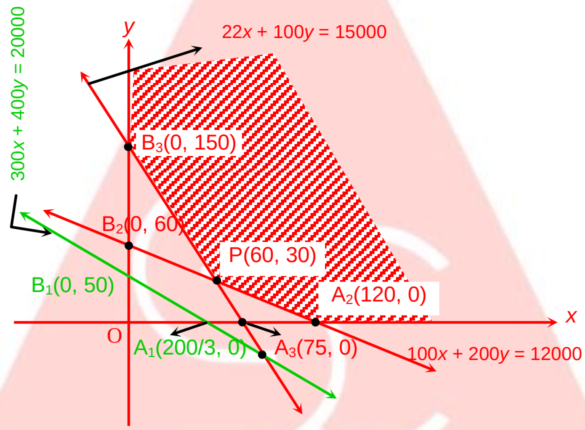
Refinery	High-grade	Medium-grade	Low-grade	Cost per day
A	100	300	200	Rs 400
B	200	400	100	Rs 300
Minimum requirement	12,000	20,000	15,000	

Suppose refineries *A* and *B* should run for *x* and *y* days respectively to minimize the total cost.
The mathematical form of the above LPP is

Minimize $Z = 400x + 300y$

Subject to $100x + 200y \ge 12000$, $300x + 400y \ge 20000$, $200x + 100y \ge 15000$ and $x, y \ge 0$

The feasible region of the above LPP is represented by the shaded region in figure.



The corner points of the feasible region are $A_2(120, 0)$, $P(60, 30)$ and $B_3(0, 150)$

The value of the objective function at these points are given in the following table:

Point (x, y)	Value of the objective function $Z = 400x + 300y$
$A_2(120, 0)$	$Z = 400 \times 120 + 300 \times 0 = 48000$
$P(60, 30)$	$Z = 400 \times 60 + 300 \times 30 = 33000$
$B_3(0, 150)$	$Z = 400 \times 0 + 300 \times 150 = 45000$

Clearly, *Z* is minimum when *x* = 60, *y* = 30.

Hence, the machine *A* should run for 60 days and the machine *B* should run for 30 days to minimize the cost while satisfying the constraints.

Illustration 6

Question: **There are two factories located one at place *P* and the other at place *Q*. From these locations, a certain commodity is to be delivered to each of the three depots situated at *A, B* and *C* . The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at *P* and *Q* are respectively 8 and 6 units. The cost of transportation per unit is given below:**

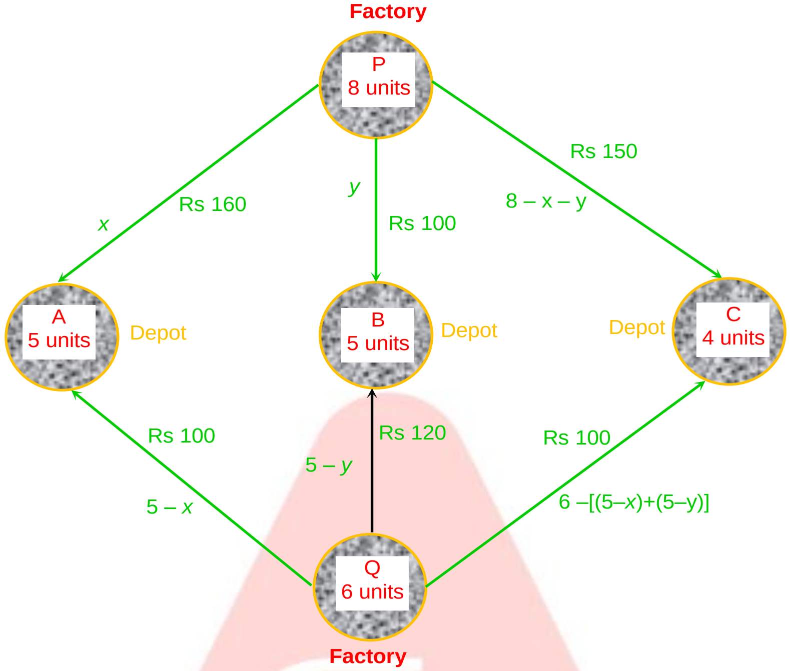
From/ To	Cost (in Rs)		
	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

Solution: The problem can be explained diagrammatically as follows:

Let *x* units and *y* units of the commodity be transported from the factory at *P* to the depots at *A* and

B respectively. Then $(8-x-y)$ units will be transported to depot at C.



Hence, we have $x \ge 0$, $y \ge 0$ and $8-x-y \ge 0$ i.e., $x \ge 0$, $y \ge 0$ and $x+y \le 8$

Now, the weekly requirement of the depot at A is 5 units of the commodity. Since x units are transported from the factory at P, the remaining $(5-x)$ units need to be transported from the factory at Q. Obviously, $5-x \ge 0$ i.e., $x \le 5$

Similarly, $(5-y)$ and $6-(5-x+5-y)=x+y-4$ units are to be transported from the factory at Q to the depots at B and C respectively.

Thus, $5-y \ge 0$, $x+y-4 \ge 0$ i.e., $y \le 5$, $x+y \ge 4$

Total transportation cost Z is given by

$$Z = 160x + 100y + 100(5-x) + 120(5-y) + 100(x+y-4) + 150(8-x-y)$$
$$= 10(x-7y+190)$$

Therefore, the problem reduces to

Minimize $Z = 10(x-7y+190)$

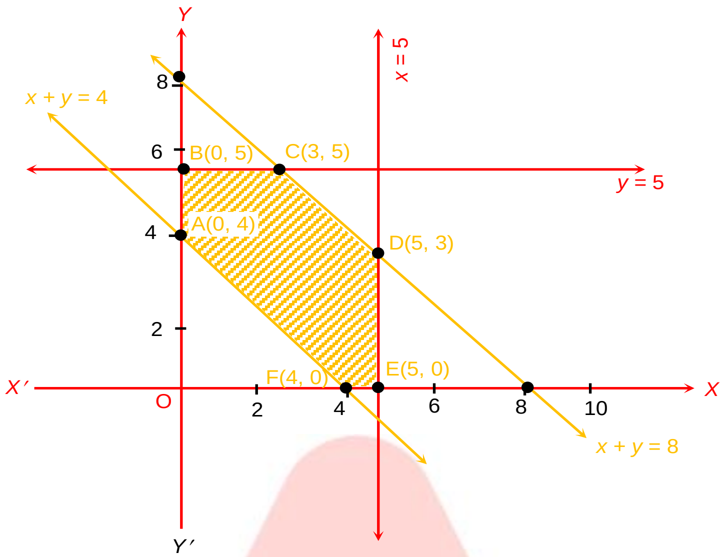
Subject to the constraints: $x \ge 0$, $y \ge 0$ …(i)

$x+y \le 8$ …(ii)

$x \le 5$ …(iii)

$y \le 5$ …(iv)

and $x+y \ge 4$ …(v)



The shaded region *ABCDEF* represented by the constraints (i) to (v) is the feasible region. Observe that the feasible region is bounded. The coordinates of the corner points of the feasible region are (0, 4), (0, 5), (3, 5), (5, 3), (5, 0) and (4, 0).
Let us evaluate *Z* at these points.

Corner Point	$Z = 10(x - 7y + 190)$
(0, 4)	1620
(0, 5)	1550 ← Minimum
(3, 5)	1580
(5, 3)	1740
(5, 0)	1950
(4, 0)	1940

From the table, we see that the minimum value of *Z* is 1550 at the point (0, 5).
Hence, the optimal transportation strategy will be to deliver 0, 5 and 3 units from the factory at *P* and 5, 0 and 1 units from the factory at *Q* to the depots at *A*, *B* and *C* respectively. Corresponding to this strategy, the transportation cost would be minimum, i.e., Rs 1550.

Illustration 7

Question: **A firm can produce three types of cloth, say C_1, C_2, C_3. Three kinds of wool are required for it, say red wool, green wool and blue wool. One unit of length C_1 needs 2 metres of red wool, 3 metres of blue wool; one unit of cloth C_2 needs 3 metres of red wool, 2 metres of green wool and 2 metres of blue wool; and one unit of cloth C_3 needs 5 metres of green wool and 4 metres of blue wool. The firm has only a stock of 16 metres of red wool, 20 metres of green wool and 30 metres of blue wool. It is assumed that the income obtained from one unit of length of cloth C_1 is Rs 6, of cloth C_2 is Rs 10 and of cloth C_3 is Rs 8. Formulate the problem as a linear programming problem to maximize the income.**

Solution: The given information can be put in the following tabular form:

	Cloth C_1	Cloth C_2	Cloth C_3	Total quality of wool available
Red wool	2	3	0	16
Green wool	0	2	5	20
Blue wool	3	2	4	30
Income (in Rs)	6	10	8	

Let x_1, x_2 and x_3 be the quantity produced in metres of the cloth of type C_1, C_2 and C_3 respectively.
Since 2 metres of red wool are required for one metre of cloth C_1 and x_1 metres of cloth C_1 are produced, therefore $2x_1$ metres of red wool will be required for cloth C_1. Similarly, cloth C_2 requires $3x_2$ metres of red wool and cloth C_3 does not required red wool. Thus,

the total quantity of red wool required is $2x_1 + 3x_2 + 0x_3$.

But the maximum available quantity of red wool is 16 metre.

$\therefore \quad 2x_1 + 3x_2 + 0x_3 \le 16$

Similarly, the total quantities of green and blue wool required are $0x_1 + 2x_2 + 5x_3$ and $3x_1 + 2x_2 + 4x_3$ respectively.

But the total quantities of green and blue wool available are 20 metres and 30 metres respectively.

$\therefore \quad 0x_1 + 2x_2 + 5x_3 \le 20$ and $3x_1 + 2x_2 + 4x_3 \le 30$

Also, we cannot produce negative quantities, therefore

$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$

The total income is $Z = 6x_1 + 10x_2 + 8x_3$

Hence, the linear programming problem for the given problem is

Maximize $Z = 6x_1 + 10x_2 + 8x_3$

Subject to the constraints:

$2x_1 + 3x_2 + 0x_3 \le 16,\ 0x_1 + 2x_2 + 5x_3 \le 20,\ 3x_1 + 2x_2 + 4x_3 \le 30$

and $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$

Illustration 8

Question: **Determine graphically the minimum value of the objective function**

$\boldsymbol{Z = -50x + 20y}$ **...(i)**

subject to the constraints:

$\boldsymbol{2x - y \ge -5}$ **...(ii)**

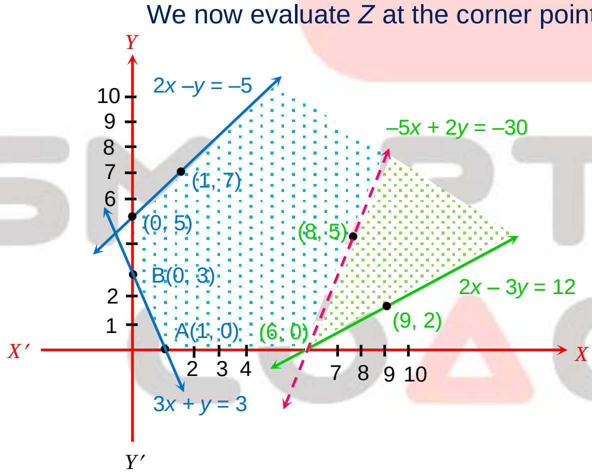
$\boldsymbol{3x + y \ge 3}$ **...(iii)**

$\boldsymbol{2x - 3y \le 12}$ **...(iv)**

$\boldsymbol{x \ge 0,\ y \ge 0}$ **...(v)**

Solution: First of all, let us graph the feasible region of the system of inequalities (ii) to (v). The feasible region (shaded) is shown in the figure. Observe that the feasible region is **unbounded.**

We now evaluate Z at the corner points.



Corner Point	$Z = -50x + 20y$
(0, 5)	100
(0, 3)	**60**
(1, 0)	**–50**
(6, 0)	**–300 ← smallest**

From this table, we find that –300 is the smallest value of Z at the corner point (6, 0). Can we say that minimum value of Z is –300?

Note that if the region would have been bounded, this smallest value of Z is the minimum value of Z (by theorem2). But here we see that the feasible region is unbounded. Therefore –300 may or may not be the minimum value of Z. To decide this issue, we graph the inequality.

$-50x + 20y < -300$

i.e., $-5x + 2y < -30$

and check whether the resulting open half plane has points in common with feasible region or not. If it has common points, then –300 will not be the minimum value of Z.

Otherwise, –300 will be the minimum value of Z.

As shown in the figure, it has common points. Therefore, $Z = -50 + 20y$ has no minimum value subject to the given constraints.

Illustration 9

Question: **Minimize $Z = 3x + 2y$**

Subject to the constraints: $x + y \geq 8$ …(i)

$3x + 5y \leq 15$ …(ii)

$x \geq 0,\ y \geq 0$ …(iii)

Solution: Let us graph the inequalities (i) to (iii). Is there any feasible region? Why is so?

From the figure, there is no point satisfying all the constraints simultaneously. Thus, the problem is having no feasible region and hence no feasible solution.

Remarks:

From the examples which we have discussed so far, we notice some general features of linear programming problems:

(i) The feasible region is always a convex region.

(ii) The maximum (or minimum) solution of the objection of the objective function occurs at the vertex (corner) of the feasible region. If two corner points produce the same maximum (or minimum) value of the objective function, then every point on the line segment joining these points will also give the same maximum (or minimum) value.

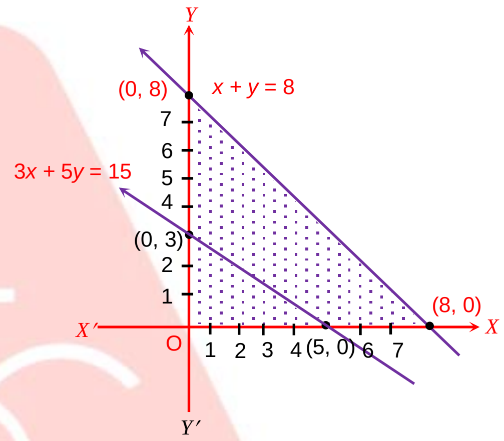


Illustration 10

Question: **Every gram of wheat provides 0.1 gm of proteins and 0.25 gm of carbohydrates. The corresponding values of rice are 0.05 gm and 0.5 gm respectively. Wheat costs Rs. 4 per kg and rice Rs. 6. The minimum daily requirements of proteins and carbohydrates for an average child are 50 gms and 200 gms respectively. In what quantities should wheat and rice be mixed in the daily diet to provide minimum daily requirements of proteins and carbohydrates at minimum cost.**

Solution: Suppose x gms of wheat and y grams of rice are mixed in the daily diet.

Since every gram of wheat provides 0.1 gm of proteins and every gram of rice gives 0.05 gm of proteins. Therefore, x gms of wheat and y grams of rice will provide $0.1x + 0.05y$ gms of proteins.

But the minimum daily requirement of proteins is of 50 gms

$$\therefore \quad 0.1x + 0.05y \geq 50 \quad \Rightarrow \quad \frac{x}{10} + \frac{y}{20} \geq 50$$

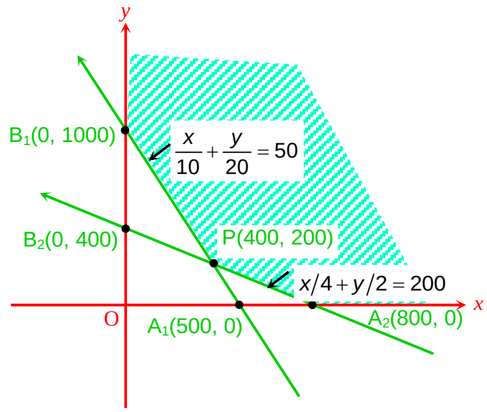
Similarly, x gms of wheat and y gms of rice will provide $0.25x + 0.5y$ gms of carbohydrates and the minimum daily requirement of carbohydrates is of 200 gms

$$\therefore \quad 0.25x + 0.5y \geq 200 \quad \Rightarrow \quad \frac{x}{4} + \frac{y}{2} \geq 200$$

Since the quantities of wheat and rice cannot be negative. Therefore, $x \geq 0,\ y \geq 0$

It is given that wheat costs Rs 4 per kg and rice Rs 6 per kg.

So, x gms of wheat and y gms of rice will cost Rs $\frac{4x}{1000} + \frac{6y}{1000}$



Hence, the given linear programming problem is

Minimize $Z = \frac{4x}{1000} + \frac{6y}{1000}$

Subject to the constraints: $\frac{x}{10} + \frac{y}{20} \ge 50,\ \frac{x}{4} + \frac{y}{2} \ge 200$ and $x \ge 0,\ y \ge 0$

The solution set of the linear constraints is shaded in figure. The vertices of the shaded region are $A_2(800, 0)$, $P(400, 200)$ and $B_1(0, 1000)$.

The values of the objective function at these points are given in the following table:

Point (x_1, x_2)	Value of objective function $Z = \frac{4x}{1000} + \frac{6y}{1000}$
$A_2(800, 0)$	$Z = \frac{4}{1000} \times 800 + \frac{8}{1000} \times 0 = 3.2$
$P(400, 200)$	$Z = \frac{4}{1000} \times 400 + \frac{6}{1000} \times 200 = 2.8$
$B_1(0, 1000)$	$Z = \frac{4}{1000} \times 0 + \frac{6}{1000} \times 1000 = 6$

Clearly, Z is minimum for x = 400, y = 200 and the minimum value of Z is 2.8.
Hence, the diet cost is minimum when x = 400 and y = 200. The minimum diet cost is Rs. 2.8.