Mathematical Reasoning

1 MATHEMATICAL LOGIC

The dictionary meaning of the word ‘Logic' is “the science of reasoning”. Logic is the study and analysis of the nature of valid arguments. It is the reasoning tool by which philosophers and mathematicians draw valid inferences from a given set of acts or premises. In fact, in any study, reasoning logic has a role to play. In the process of reasoning we communicate our ideas or thoughts with the help of sentences in a particular language. The following types of sentences are normally used in our every day communication:

Assertive Sentence

A sentence that makes an assertion is called an assertive sentence or a declarative sentence.

For example. “Mars supports life” is an assertive or a declarative sentence. “Any two individuals are always related” is also a declarative sentence.

Imperative Sentence

A sentence that expresses a request or a command is called an imperative sentence.

For example. “Please bring me a cup of tea” is an imperative sentence.

Exclamatory Sentence

A sentence that expresses some strong feelling is called an exclamatory sentence.

For example, “How big is the whale fish !” is an exclamatory sentence.

Interrogative Sentence

A sentence that asks some question is called an interrogative sentence.

For example, “What is your age?” is an interrogative sentence.

In this chapter, we shall be discussing about a specific type of sentences which will be called as statements or propositions.

2 STATEMENTS OR PROPOSITIONS

2.1 DEFINITION

A statement or a proposition is an assertive (or a declarative) sentence which is either true or false but not both.

A statement is assumed to be either true or false. A true statement is also known as a valid statement. If a statement is false, we say that it is an invalid statement. A statement cannot be both true and false at the same time.

A sentence which is both true and false simultaneously is not a statement, rather it is a paradox.

Consider the following sentences:

(i) Three plus four is 6.
(ii) The earth is a star.
(iii) Every rectangle is a square.
(iv) New Delhi is in Nepal.
(v) Every relation is a function

Each of these sentences is a false declarative sentence and hence each of them is a statement.

2.2 NEGATION OF A STATEMENT

The denial of a statement is called the negation of the statement.

Let us consider the statement:

P : New Delhi is a city

The negation of this statement is

It is not the case that New Delhi is a city

This can also be written as

It is false that New Delhi is a city

This can simply be expressed as

New Delhi is not a city

Definition:

If *p* is a statement, then the negation of *p* is also a statement and is denoted by ~*p* and read as 'not *p*'.

Note: While forming the negation of statement, phrases like, "It is not the case" or "It is false that" are also used.

2.3. COMPOUND STATEMENTS

If a statement is combination of two or more simple statements, then it is said to be a compound statement or a compound proposition.

The statement "The school works or a holiday is declared" is a compound statement as it is a combination of the statements: "The school woks" and "A holiday is declared".

3 BASIC LOGICAL CONNECTIVES OR LOGICAL OPERATORS

The phrases or words which connect simple statements are called logical connectives or sentential connectives or simply connectives or logical operators.

In the following table, we list some possible connectives, their symbols and the nature of the compound statement formed by them.

Connective	Symbol	Nature of compound statement formed by using the connective
and	$\wedge$	Conjunction
or	$\vee$	Disjunction
If ……then	$\Rightarrow$ or $\rightarrow$	Implication or conditional
If and only if (iff)	$\Leftrightarrow$ or $\leftrightarrow$	Equivalence or bi-conditional
not	$\sim$ or $\neg$	Negation

Remark: Negation is called a connective although it does not combine two or more statements.

3.1 THE WORD "AND"

Let us look at a compound statement with "And".

P : A point occupies a position and its location can be determined.

The statement can be broken into two component statements as

q: A point occupies a position.

r: Its location can be determined.

Here, we observe that both statement are true.

Let us look at another statement.

p: 42 is divisible by 5, 6 and 7

This statement has following component statements

q: 42 is divisible by 5.

r: 42 is divisible by 6.

s: 42 is divisible by 7.

Here, we know that the first is false while the other two are true.

(i) A compound statement is true only if both statements connected with 'AND' are true. Otherwise it is false.

(ii) ~ (*p* and *q*) = ~*p* or ~*q*

3.2 THE WORD "OR"

Let us look at the following statement.

p: Two lines in a plane either intersect at one point or they are parallel.

We know that this is a true statement. What does this mean? This means that if two lines in a plane intersect, then they are not parallel. Alternatively, if the two lines are not parallel, then they intersect at a point. That is, this statement is true in both the situations.

p: An ice cream or Pepsi is available with a Thali in a restaurant.

This means that a person who does not want ice cream can have a Pepsi along with Thali or one does not want Pepsi can have an ice cream along with Thali. This is called an **exclusive "Or"**.

(i) A compound statement is true if at least one of the statements connected with 'Or' is true.

(ii) $\sim(p \text{ or } q) = \sim p$ and $\sim q$

3.3 IMPLICATIONS

In this Section, we shall discuss the implications of "if-then", "only if" and "if and only if"

The statements with "if-then" are very common in mathematics. For example, consider the statement.

r: If you are born in some country, then you are a citizen of that country.

When we look at this statement, we observe that it corresponds to two statements *p* and *q* given by

p: you are born in some country.

q: you are citizen of that country.

Then the sentence "if *p* then *q*" says that in the event if *p* is true, then *q* must be true.

One of the most important facts about the sentence "if *p* then *q*" is that it does not say any thing (or phaces no demand) on *q* when *p* is false. For example, if you are not born in the country, then you cannot say anything about q. To put it in other words" not happening of *p* has no effect on happening of *q*.

Another point to be noted for the statement "if *p* then *q*" statements does not imply that *p* happens.

Then, if *p* then *q* is the same as the following:

1. '*p* implies *q*' is denoted by $p \Rightarrow q$. The symbol $\Rightarrow$ stands for implies.
   This says that 'A number is a multiple of 9 implies that it is a multiple of 3'.
2. *p* is a sufficient condition for *q*.
   This says that 'Knowing that a number as a multiple of 9 is sufficient to conclude that it is a multiple of 3'.
3. *p* only if *q*.
   This says that 'A number is a multiple of 9 only if it is a multiple of 3'.
4. *q* is a necessary condition for *p*.
   This says that 'When a number is a multiple of 9, it is necessarily a multiple of 3'.
5. $\sim q$ implies $\sim p$.
   This says that 'If a number is not a multiple of 3, then it is not a multiple of 9'.
6. If *p* and *q* are both true, then $p \Rightarrow q$ is also true.
   If *p* is false, *q* is true then $p \Rightarrow q$ is true.
   If *p* is true, *q* is false then $p \Rightarrow q$ is false.
   If *p* is false, *q* is false then $p \Rightarrow q$ is true.

3.4 CONTRAPOSITIVE AND CONVERSE

Contrapositive and converse are certain other statements which can be formed from a given statement with "if-then".

For example, let us consider the following "if-then" statement.

"If the physical environment changes, then the biological environment changes".

Then the contrapositive of this statement is

"If the biological environment does not change, then the physical environment does not change".

Note that both these statement convey the same meaning.

Also, the converse of the statement is 'If the biological environment changes then the physical environment changes'.

Contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$

Converse of $p \Rightarrow q$ is $q \Rightarrow p$

3.5 CONDITIONAL AND BI-CONDITIONAL STATEMENTS

In Mathematics we come across many statements of the form "if *p* then *q*" and "*p* if and only if *q*" such statements are called conditional statements. In this section, we shall discuss about such statements.

Important formulae/points

All kind of statements are important and stress should be given on the illustrations followed by each definition.