# MATRICES

## 1 DEFINITION OF A MATRIX

A system of '$mn$' numbers (real or complex) arranged in a rectangular array of $m$ rows and $n$ columns is called a matrix. This system can be arranged in any of the following patterns.

$$\left(\quad\right);\ \left[\quad\right];\ \left\|\quad\right\|$$

In general $a_{ij}$ represent the element (or entry) of $i^{th}$ row and $j^{th}$ column, so the matrix can be represented as $(a_{ij})$ or $[a_{ij}]$ or $||a_{ij}||$

## 2 ORDER OF A MATRIX

If any matrix $A$ contains '$m$' rows and '$n$' columns then $m \times n$ is termed as order of matrix.
Order is generally written as suffix of the array.
Now any matrix of order $m \times n$ will have the notation $[a_{ij}]_{m \times n}$ .
i.e. $A = [a_{ij}]_{m \times n}$ or $(a_{ij})_{m \times n}$ or $\|a_{ij}\|_{m \times n}$
it is obvious that $1 \le i \le m$ and $1 \le j \le n$

## 3 TYPES OF MATRIX

The elements which appear in the rectangular array are known as entries ; depending upon these entries, matrices are of following types:

### 3.1 ROW MATRIX

A single row matrix is called a row matrix or a row vector.
e.g. the matrix $[a_{11}\ a_{12}\ \ldots..\ a_{1n}]$ is a $1 \times n$ row matrix.

### 3.2 COLUMN MATRIX

A single column matrix is called a column matrix or a column vector.

e.g. the matrix $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \ldots \\ \ldots \\ a_{mj} \end{bmatrix}$ is a $m \times 1$ column matrix.

### 3.3 SQUARE MATRIX

If $m = n$, i.e. if the number of rows and columns of a matrix are equal say $n$, then it is called a square matrix of order $n$.

### 3.4 NULL (or zero) MATRIX

If all the elements of a matrix are equal to zero, then it is called a null matrix and is denoted by $O_{m \times n}$ or O.

### 3.5 DIAGONAL MATRIX

A square matrix in which all its elements are zero except those in the leading diagonal, is called a diagonal matrix. Thus in a diagonal matrix $a_{ij} = 0$ if $i \ne j$.
The diagonal matrices of order 2 and 3 are as follows:

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}.$$

The elements $a_{ij}$ of a matrix for which i = $j$ are called the diagonal elements of a matrix and the diagonal along which all these elements lie is called the principal diagonal or the diagonal of the matrix.

### 3.6 SCALAR MATRIX

A square matrix in which all the diagonal elements are equal and all other elements equal to zero is called a scalar matrix.

i.e. in a scalar matrix $a_{ij} = k$, for i = $j$ and $a_{ij} = 0$ for i $\neq j$. Thus $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ is a scalar matrix.

### 3.7 UNIT MATRIX OR IDENTITY MATRIX

A square matrix in which all its diagonal elements are equal to 1 and all other elements equal to zero is called a unit matrix or identity matrix.

e.g. a unit (or identity) matrix of order 2 and 3 are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

### 3.8 NEGATIVE OF A MATRIX

Let A = $[a_{ij}]_{m\times n}$ be a matrix. Then the negative of the matrix A is defined as the matrix $[-a_{ij}]_{m\times n}$ and is denoted by –A.

## 4 EQUALITY OF MATRICES

Two matrices $A$ and $B$ are said to be equal, written as $A = B$, if,

(i) they both are of the same order i.e. have the same number of rows and columns, and

(ii) the elements in the corresponding places of the two matrices are the same.

## 5 ADDITION AND SUBTRACTION OF MATRICES

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same type $m \times n$. Then their sum (or difference) $A + B$ (or $A - B$) is defined as another matrix of the same type, say $C = [c_{ij}]$ such that any element of $C$ is the sum (or difference) of the corresponding elements of $A$ and $B$.

$\therefore \quad C = A \pm B = [a_{ij} \pm b_{ij}]$

### 5.1 PROPERTIES OF MATRIX ADDITION

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $k(A + B) = kA + kB$ here $k$ is any scalar.
4. $A$ + O = O + $A$ = $A$ , here O {null matrix} will be additive identity.
5. If $A$ be a given matrix then the matrix $-A$ is the additive inverse of $A$ for $A + (-A)$ = null matrix O.
6. If $A$, $B$ and $C$ be three matrices of the same type

   then $A + B = A + C \Rightarrow B = C$ (Left Cancellation Law)

   and $B + A = C + A \Rightarrow B = C$ (Right Cancellation Law)

## 6 MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A = [a_{ij}]_{m\times n}$ be a matrix and $k$ a scalar. Then the matrix obtained by multiplying each element of matrix $A$ by $k$ is called the scalar multiple of $A$ and is denoted by $k\,A$.

## 6.1 PROPERTIES

- If $k_1$ and $k_2$ are scalars and $A$ be a matrix, then $(k_1 + k_2) A = k_1A + k_2A$.
- If $k_1$ and $k_2$ are scalars and $A$ be a matrix, then $k_1(k_2 A) = (k_1k_2)A$.
- If $A$ and $B$ are two matrices of the same order and $k$, a scalar, then $k(A + B) = kA + kB$.
  i.e. the scalar multiplication of matrices distributes over the addition of matrices.
- If $A$ is any matrix and $k$ be a scalar, then $(-k)A = -(kA) = k(-A)$.

# 7 MULTIPLICATION OF TWO MATRICES

Let $A = [a_{ij}]$ be $m \times p$ matrix and $B = [b_{ij}]$ be $p \times n$ matrix. These matrices $A$ and $B$ are such that the number of columns of $A$ are the same as the number of rows of $B$ each being equal to p. Then the product $AB$ (in the order it is written) will be a matrix $C = [c_{ij}]$ of the type $m \times n$.

Where $c_{ij}$ will be the element of C occurring in $i^{th}$ row and $j^{th}$ column and it will be row by column product of $i^{th}$ row of A having p columns with $j^{th}$ column of $B$ having $p$ rows, the elements of which are

$a_{i1}\ a_{i2}\ \ldots\ a_{ip}$ and $b_{1j}$
$a_{i1}\ a_{i2}\ \ldots\ a_{ip}$ and $b_{2j}$
……………………
……………………

$$\therefore \quad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

The summation is to be performed w.r.t. repeated suffix $k$.

Above gives us the particular $i$-$j$th element of C which is $m \times n$ type. For getting an element of C occurring in 2$^{nd}$ row and 3$^{rd}$ column we shall put $i = 2$ and $j = 3$.

$$\therefore \quad c_{23} = \sum_{k=1}^{p} a_{2k}b_{k3} = a_{21}b_{13} + a_{22}b_{23} + \ldots\ldots a_{2p}b_{p3}$$

There being $m$ rows in $A$, $i$ can take values from 1 to $m$ and there being $n$ columns in $B$, $j$ can take values from 1 to $n$, and thus we shall get all the $mn$ elements of $C$.

$$\text{Again} \quad c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj} \qquad \ldots\text{(i)}$$

Above gives us $i$-$j$th element of $AB$ which is of $m \times n$ type having $m$ rows and $n$ columns.

## 7.1 ELEMENTS OF *j*th COLUMN OF AB

For getting elements of $j$th column, $j$ will remain fixed for $j$th column whereas $i$ will change from 1 to $m$ as there are $m$ rows in $AB$.

Hence giving $i$ the values 1, 2, 3……$m$ and keeping $j$ fixed in (i) we shall get all the elements of $j$th column of $AB$.

$$\therefore \quad j\text{th column of } AB \text{ is } \sum_{k=1}^{p} a_{1k}\ b_{kj},\ \sum_{k=1}^{p} a_{2k}b_{kj},\ \ldots\ldots,\ \sum_{k=1}^{p} a_{mk}b_{kj}$$

## 7.2 AN EASY WAY TO REMEMBER

If we denote the ordered set of rows of $A$ by $R_1$, $R_2$, $R_3$ each having 2 elements and ordered set of columns of $B$ by $C_1$, $C_2$, each having 2 elements.

$$\text{Then} \quad AB = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}_{3\times1} [C_1 C_2]_{1\times2} = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \\ R_3C_1 & R_3C_2 \end{bmatrix}_{3\times2}$$

## 7.3 FEW IMPORTANT THINGS FOR THE MULTIPLICATION

1. **Condition for product AB to exist or to be defined:** If $A$ and $B$ be two matrices then their product is defined or in other words $A$ is *conformable* to $B$ for multiplication if the number of columns of $A$ is the same as the number of rows in $B$. i.e. If $A$ be $m \times p$ and $B$ be $p \times n$, the

matrix $AB$ will be of the type $m \times n$.

2. **Pre-multiplication and post multiplication**

When we say multiply $A$ by $B$ then it could mean both $AB$ or $BA$ where $A$ and $B$ are any numbers. But when $A$ and $B$ are matrices then as seen above $AB$ and BA do not necessarily mean the same thing. If $AB$ is defined for matrix multiplication $BA$ may not be defined. To avoid this when we say product $AB$ it would mean the matrix $A$ post-multiplied by $B$ and when we say product $BA$ it would mean matrix $A$ pre-multiplied by $B$. In $AB$, $A$ is called *prefactor* and $B$ *post factor*.

3. In the case when both $A$ and $B$ are square matrices of the same type then also both $AB$ and $BA$ are defined and the product matrix is also a matrix of the same type but still $AB \neq BA$.
4. Again we know that when $ab = 0$ it means that either $a$ or $b$ (or both) is zero. But $AB = O$ i.e. a null matrix does not necessarily imply that either $A$ or $B$ = O as shown above because neither $A$ nor $B$ is null matrix whereas $AB$ is a null matrix.

## 8 OPERATIONS REGARDING MATRICES

### 8.1 TRANSPOSE OF A MATRIX

If $A$ be a given matrix of the type $m \times n$ then the matrix obtained by changing the rows of $A$ into columns and columns of $A$ into rows is called transpose of matrix $A$ and is denoted by $A'$ or $A^T$. As there are $m$ rows in $A$ therefore there will be $m$ columns in $A'$ and similarly as there are $n$ columns in $A$ there will be $n$ rows in $A'$.

**Properties of transpose**

(i) $(A')' = A$

(ii) $(KA)' = KA'$. $K$ being a scalar.

(iii) $(A \pm B)' = A' \pm B'$

(iv) $(AB)' = B'A'$.

(v) $(ABC)' = C'B'A'$.

### 8.2 THE CONJUGATE OF A MATRIX

Let $A = [a_{ij}]$ be a given matrix then the matrix obtained by replacing all the elements by their conjugate complex is called the conjugate of matrix $A$ and will be represented by $\overline{A}$ i.e. $\overline{A} = [\overline{a}_{ij}]$.

(i) $\overline{\overline{A}} = A$

(ii) $\left(\overline{A+B}\right) = \overline{A} + \overline{B}$

(iii) $(\overline{AB}) = \overline{A}\,\overline{B}$

### 8.3 TRANSPOSE OF THE CONJUGATE OF A MATRIX

Transpose of the conjugate of a matrix is equal to the conjugate of the transpose of a matrix $A$ i.e. $(\overline{A})' = (\overline{A'})$ and written as $A^{\theta}$.

(i) $(A^{\theta})^{\theta} = A$

(ii) $(A + B)^{\theta} = A^{\theta} + B^{\theta}$

(iii) $(KA)^{\theta} = \overline{K}A^{\theta}$, $K$ being a scalar

(iv) $(AB)^{\theta} = B^{\theta} A^{\theta}$

### 8.4 MINOR OF ANY ELEMENT OF A MATRIX

Consider the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

If we leave the row and the column passing through the element $a_{ij}$, then the second order determinant thus obtained is called the minor of the element $a_{ij}$ and we shall denote it by $M_{ij}$. In this way we can get 9 minors corresponding to the 9 elements of $\Delta$.

For example:

the minor of the element $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$ ,

the minor of the element $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$ ,

the minor of the element $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11}$ , and so on.

### 8.5 COFACTOR OF ANY ELEMENT OF A MATRIX

The minor $M_{ij}$ multiplied by $(-1)^{i+j}$ is called cofactor of the element $a_{ij}$. We shall denote the cofactor of an element by the $C_{ij}$. With this notation, cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$.

### 8.6 DETERMINANT OF ANY MATRIX

If matrix $A = [a_{ij}]$ is a square matrix of order '$n$' then

determinant of A = $\left(\sum_{k=1}^{n} a_{1k}C_{1k}\right) = \left(\sum_{k=1}^{n} a_{2k}C_{2k}\right) = \ldots\ldots\ldots = \ldots\ldots\ldots$

$= \left(\sum_{k=1}^{n} a_{k1}C_{k1}\right) = \left(\sum_{k=1}^{n} a_{k2}C_{k2}\right) = \ldots\ldots\ldots = \ldots\ldots\ldots$

here $C_{ik}$ represents cofactor of the element of $i^{th}$ row and $k^{th}$ column of matrix A.

for $3 \times 3$ order matrix $A$ ; det $A$ (or $|A|$) $= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

$= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$

$= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$

$= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

### 8.7 TRACE OF A MATRIX

Let A be a square matrix of order $n$. The sum of the elements of A lying along the principal diagonal is called the trace of $A$. We shall write the trace of $A$ as $tr\ A$. Thus if $A = [a_{ij}]_{n\times n}$, then $tr\ A = \sum_{i=1}^{n} a_i = a_{11} + a_{22} + \ldots\ldots + a_{nn}$ .

## 9 TYPES OF MATRIX ON THE BASIS OF OPERATIONS

### 9.1 SYMMETRIC MATRIX

A square matrix $A = [a_{ij}]$ is said to be symmetric if its $(i, j)^{th}$ element is the same as its $(j, i)$th element i.e., if $a_{ij} = a_{ji}$ for all $i$, $j$.

### 9.2 SKEW SYMMETRIC MATRIX

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if the $(i, j)^{th}$ element of $A$ is the negative of the $(j, i)$th element of A i.e., if $a_{ij} = -a_{ji}$ for all $i$, $j$.

### 9.3 HERMITIAN MATRIX

A square matrix $A = [a_{ij}]$ is said to be Hermitian if the $(i, j)^{th}$ element of A is equal to conjugate complex of the $(j, i)$th element of A i.e., if $a_{ij} = \overline{a}_{ji}$ for all $i$ and $j$.

### 9.4 SKEW HERMITIAN MATRIX

A square matrix $A = [a_{ij}]$ is said to be Skew Hermitian if the $(i, j)^{th}$ element of A is equal to the negative of conjugate complex of the $(j, i)$th element of A i.e., if $a_{ij} = -\bar{a}_{ji}$ for all i and $j$.

### 9.5 ORTHOGONAL MATRIX

A square matrix $A$ is said to be orthogonal if $A'A = I = AA'$

### 9.6 UNITARY MATRIX

A square matrix $A$ is said to be unitary if $A^{\theta}A = I = AA^{\theta}$

### 9.7 IDEMPOTENT MATRIX

A matrix such that $A^2 = A$ is called idempotent matrix.

### 9.8 NILPOTENT MATRIX

A matrix $A$ will be called a nilpotent matrix if $A^k = O$ (null matrix) where $k$ is a positive integer. If however $k$ is the least positive integer for which $A^k = O$ then $k$ is the *index* of the nilpotent matrix $A$.

### 9.9 INVOLUTRY MATRIX

A matrix $A$ such that $A^2 = I$ is called involutry matrix.

## 10 ADJOINT OF A SQUARE MATRIX

Let $A = [a_{ij}]_{n\times n}$ be any $n \times n$ matrix. The transpose $B'$ of the matrix $B = [C_{ij}]_{n\times n}$, where $C_{ij}$ denotes the cofactor of the element $a_{ij}$ in the determinant $|A|$, is called the adjoint of the matrix $A$ and is denoted by the symbol *adj A*.

## 11 INVERSE OF A MATRIX

Let $A$ be any $n$-rowed square matrix. Then a matrix $B$, if it exists, such that $AB = BA = I_n$ is called inverse of A.

The necessary and sufficient condition for a square matrix $A$ to possess the inverse is that $|A| \neq 0$.

If $A$ be an invertible matrix, then the inverse of $A$ is $\frac{1}{|A|}$ Adj. $A$. It is usual to denote the inverse of $A$ by $A^{-1}$.

- **Theorem (Uniqueness of inverse)**

Inverse of a square matrix if it exists is unique.

**Proof:**

Let $A = [a_{ij}]_{n\times n}$ be a square matrix. Let inverse of $A$ exist.

**To prove inverse of $A$ is unique:**

If possible, let $B$ and $C$ be two inverses of $A$

Then $AB = BA = I_n$

And $AC = CA = I_n$

Now $B = BI_n$

$= B(AC)$ $[\because AC = I_n]$

$= (BA)C = I_nC = C$

Hence $B = C \Rightarrow$ inverse of $A$ is unique

### 11.1 PROPERTIES

(i) $(AB)^{-1} = B^{-1}A^{-1}$ ,

(ii) $(A')^{-1} = (A^{-1})'$
(iii) $(A^{-1})^{\theta} = (A^{\theta})^{-1}$

## 13 ELEMENTARY OPERATIONS OR ELEMENTARY TRANSFORMATIONS OF A MATRIX

**Definition:**

Any of the following operations is called an **elementary transformation (operation).**

(i) The interchange of any two rows (or columns).
(ii) The multiplication of the elements of any row (or column) by a non-zero number.
(iii) The addition to the elements of any row (or column), the corresponding elements of any other row (or column) multiplied by a non-zero number.

Any elementary transformation is called a row transformation or column transformation according as it applies to rows or columns.

Clearly, there will be a total of six elementary operations (transformations) on a matrix, three of them are due to rows and are called row operations where as three of them are due to columns and are called column operations.

1. The elementary operations of interchange of *i*th row and *j*th row is denoted by $R_i \leftrightarrow R_j$ and interchange of *i*th column and *j*th column is denoted by $C_i \leftrightarrow C_j$.

**Example:**

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$ i.e., interchanging Ist row and 3rd row matrix *A* becomes the matrix

$$B = \begin{bmatrix} 2 & 0 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

2. The elementary operation of multiplication of the elements of the *i*th row by a non-zero number *k* is denoted by $R_i \to kR_i$.

Similarly, the multiplication of the elements of the *i*th column by a non-zero number *k* is denoted by $C_i \to kC_i$.

**Example:**

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$

On multiplying the elements of 3rd column of matrix *A* by 2, i.e., on applying $C_3 \to 2C_3$, we get the new matrix

$$B = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 3 & 8 \\ 2 & 0 & 10 \end{bmatrix}$$

3. The elementary operation of the addition to the elements of the *i*th row, the corresponding elements of the *j*th row multiplied by a non-zero number *k* is denoted by $R_i \to R_i + kR_j$.

Similarly, the elementary operation of the addition to the elements of the *i*th column, the corresponding elements of the *j*th column multiplied by a non-zero number *k* is denoted by $C_i \to C_i + kC_j$.

**Example:**

Let $A=\begin{bmatrix}1&2&3\\2&3&4\\2&0&5\end{bmatrix}$ , $B=\begin{bmatrix}1&4&3\\2&7&4\\2&4&5\end{bmatrix}$

On applying the elementary operation $C_2 \to C_2 + 2C_1$, matrix *A* becomes the matrix *B*.

- **Equivalent Matrices**

Two matrices *A* and *B* are said to be **equivalent** if one can be obtained from other by applying a finite number of elementary operations on the other matrix. If *A* and *B* are equivalent matrices we write $A \sim B$.

**Example:**

Let $A=\begin{bmatrix}1&2&3\\2&3&4\\2&0&5\end{bmatrix}$ and $B=\begin{bmatrix}2&3&8\\1&2&6\\2&0&10\end{bmatrix}$

Now $A=\begin{bmatrix}1&2&3\\2&3&4\\2&0&5\end{bmatrix} \sim \begin{bmatrix}2&3&4\\1&2&3\\2&0&5\end{bmatrix}$ [Applying $R_1 \leftrightarrow R_2$]

$\sim \begin{bmatrix}2&3&8\\1&2&6\\2&0&10\end{bmatrix} = B$ [Applying $C_3 \to 2C_3$]

Here $A \sim B$ as *B* has been obtained from *A* by applying two elementary operations.

- **Elementary Matrix:**

A matrix obtained from unit matrix by a single elementary operation is called an elementary matrix.

**Example:**

$I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Then $A=\begin{bmatrix}2&0\\0&1\end{bmatrix}$ $[R_1 \to 2R_1]$

is an elementary matrix.

**Inverse of a matrix by elementary operations (Elementary operations on matrix equation)**

Let *A, B* and *X* be three matrices of the same order such that

$X = AB$ …(i)

The matrix equation (i) will be also valid if we apply a row operation on matrix *X* [occurring on the L.H.S. of equation (i)] and the same row operation on matrix *A* (the first factor of product *AB* on the matrix on R.H.S.)

Thus on the application of a sequence of row operations on the matrix equation *X = AB* (these row operations are applied on *X* and on the first matrix *A* of product *AB* simultaneously), the matrix equation is still valid (we assume this fact without proof).

Similarly a sequence of elementary column operations on the matrix equation *X = AB* can be applied simultaneously on *X* and on the second matrix *B* of product *AB* and the equation will be still valid.

In view of the above mentioned fact, it is clear that we can find the inverse of a matrix *A*, if it exists, by using either a sequence of elementary row operations or a sequence of elementary column operations but not both simultaneously.

**Using row operation**

Apply a series of row operation on *A = IA* till we get *I = BA*

Now by definition of inverse of a matrix $B = A^{-1}$

**Using Column operation**

Apply a series of column operations on $A = AI$ till we get $I = AB$. By definition of inverse $B$ is inverse of $A$.

## 14 RANK OF A MATRIX

*A number r is said to be the rank of a matrix A if it possesses the following two properties:*

(i) There is at least one square submatrix of $A$ of order $r$ whose determinant is not equal to zero.

(ii) If the matrix $A$ contains any square submatrix of order $r + 1$, then the determinant of every square submatrix of A of order $r + 1$, should be zero.

In short the rank of a matrix is the order of any highest order non-vanishing minor of the matrix.

## 15 HOMOGENEOUS LINEAR EQUATIONS

The equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots.. + a_{1n}x_n &= 0\ , \\ a_{21}x_1 + a_{22}x_2 + \ldots.. + a_{2n}x_n &= 0\ , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots... + a_{mn}x_n &= 0 \end{aligned}\right\} \quad \ldots\text{(i)}$$

represents system of $m$ homogeneous equations in $n$ unknowns $x_1$, $x_2$ , ……. , $x_n$. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{31} & a_{32} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix}_{n\times 1}, \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \end{bmatrix}_{m\times 1},$$

where $A$, $X$, O are $m \times n$, $n \times 1$, $m \times 1$ matrices respectively. Then obviously we can write the system of equations (i) in the form of a single matrix equation

$$AX = O \quad \ldots\text{(ii)}$$

The matrix $A$ is called the coefficient matrix of the system of equations.

(i) If $|A| = 0$ the system has infinitely many solutions.

(ii) If $|A| \neq 0$ the system has zero solution or trivial solutions.

**These conclusions can also be written on the basis of rank method as follows:**

Suppose we have $m$ equations in $n$ unknowns. Then the coefficient matrix $A$ will be of the type $m \times n$. Let $r$ be the rank of the matrix $A$. Obviously $r$ cannot be greater than $n$ (the number of columns of the matrix $A$). Therefore we have either $r = n$ or $r < n$.

**Case I:** If $r = n$, the equation $AX$ = O will have $n - n$ i.e., no linearly independent solutions. In this case the zero solution will be the only solution. We know that zero vector forms a linearly dependent set.

**Case II:** If $r < n$, we shall have $n - r$ linearly independent solutions. Any linear combination of these $n - r$ solutions will also be a solution of $AX$ = O. Thus in this case the equation $AX$ = O will have an infinite number of solutions.

**Case III:** Suppose $m < n$ i.e., the number of solutions is less than the number of unknowns. Since $r \leq m$, therefore $r$ is definitely less than $n$. Hence in this case the given system of equations must posses a non-zero solution. The number of solutions of the equation $AX$ = O will be infinite.

## 15 SYSTEM OF LINEAR NON HOMOGENEOUS EQUATIONS

The equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots.. + a_{1n}x_n = b_1 \;, \\ a_{21}x_1 + a_{22}x_2 + \ldots.. + a_{2n}x_n = b_2 \;, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m \end{array}\right\} \quad \ldots(i)$$

be a system of $m$ non-homogeneous equations in $n$ unknowns $x_1, x_2, \ldots\ldots, x_n$. If we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}_{n\times 1}, B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}_{m\times 1},$$

where $A$, $X$, $B$ are $m \times n$, $n \times 1$, $m \times 1$ matrices respectively the above equations can be written in the form of a single matrix equation $AX = B$.

Any set of values of $x_1, x_2, \ldots.., x_n$ which simultaneously satisfy all these equations is called a solution of the system (i). When the system of equations has one or more solutions, the equations are said to be consistent, otherwise they are said to be inconsistent.

If $B \neq 0$ the system (i) is said to be non-homogenous.

(i) If $|A| \neq 0 \quad \Rightarrow \quad X = A^{-1}B$, where $A^{-1} = \dfrac{Adj\ A}{|A|}$

the given system has unique solution.

(ii) If $|A| = 0$

$\because \quad AX = B$

$\Rightarrow \quad (adj\ A)AX = (adj\ A)B \quad \Rightarrow \quad |A|\ X = (adj\ A)B$

$\Rightarrow \quad (adj\ A)B = 0 \quad [\because |A| = 0]$

which is true for infinite values of $X$.

$\therefore$ for infinitely many solutions to the system $(adj\ A)B = 0$

Clearly for no solution $(adj\ A)B \neq 0$

**These conclusions can also be written on the basis of rank method as follows:**

The matrix $[A\ B] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$

is called the augmented matrix of the given system of equations.

Suppose the coefficient matrix A is of the type $m \times n$, i.e., we have $m$ equations in $n$ unknowns. Write the augmented matrix [$A$ $B$] and reduce it to a Echelon form by applying only $E$-row transformations and comparing the ranks of the augmented matrix [$A$ $B$] and the coefficient matrix $A$. Then the following different cases arise:

**Case I:** Rank $A$ < Rank [$A$ $B$]

In this case the equations $AX = B$ are inconsistent i.e., they have no solution.

**Case II:** Rank $A$ = Rank [$A$ $B$] = $r$ (say).

In this case the equations $AX = B$ are consistent i.e., they possess a solution. If $r < m$, then in the process of reducing the matrix [$A$ $B$] to Echelon form, ($m - r$) equations will then the replaced by an equivalent system of $r$ equations. From these $r$ equations we shall be able to express the values of some $r$ unknowns in terms of the remaining $n - r$ unknowns which can be given any arbitrary chosen values.

If $r = n$, then $n - r = 0$, so that no variable is to be assigned arbitrary values and therefore in this case there will be a unique solution.

If $r < n$ , then $n - r$ variables can be assigned arbitrary values. So in this case there will be an infinite number of solutions Only $n - r + 1$ solutions will be linearly independent and the rest of the solutions will be linear combinations of them.

If $m < r$, then $r \leq m < n$. Thus in this case $n - r > 0$. Therefore when the number of equations is less than the number of unknowns, the equations will always have an infinite number of solution provided they are consistent.

For non singular matrix $A$

Equation $AX = B \Rightarrow X = A^{-1}B$

By comparing entries on both the sides we have unique solution for given system of equations.

# MIND MAP

## MATRIX

$A = [a_{ij}]_{m\times n}\ ,\ 1 \le i \le m\ ,\ 1 \le j \le n$

### MULTIPLICATION OF MATRICES

- Product AB is possible if matrix A has order $m \times n$ and B has the order $n \times l$.
- $AB = [C_{ij}]_{m\times l}$ where $C_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$ .
- $AB \ne BA$ (in general)
- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$

### KINDS OF MATRICES DEPENDING UPON THEIR ENTRIES

- Null matrix : If $a_{ij} = 0$ , $\forall$ i and j.
- Square matrix : If m = n.
- Unit matrix : Square matrix having $a_{ij} = 0$ , $\forall$ $i \ne j$ and $a_{ij} = 1$ for i = j.
- Row matrix : If m = 1
- Column matrix : If n = 1
- Diagonal matrix : Square matrix $a_{ij} = 0$ , $\forall$ $i \ne j$ and $a_{ij} \ne 0$ for i = j
- Upper triangular matrix : If $a_{ij} = 0$ for i > j.
- Lower triangular matrix : If $a_{ij} = 0$ for i < j.

### MINOR

- Minor $M_{ij}$ of any element $a_{ij}$ of the matrix A is equal to the determinant of matrix leaving $i^{th}$ row and $j^{th}$ column.

### COFACTOR

- Cofactor $C_{ij}$ of any element $a_{ij}$ of the matrix $= (-1)^{i+j}M_{ij}$

### ADJOINT OF MATRIX

- For square matrix A adj A = $[C_{ij}]^T$

### INVERSE OR RECIPROCAL OF A NON SINGULAR MATRIX A

- $(A^{-1}) = \dfrac{adj\ A}{|A|}$

### OPERATIONS RELATED TO MATRICES

- Transpose $(A^T) = [a_{ji}]_{n\times m}$
- Conjugate $(\bar{A}) = [\bar{a}_{ij}]_{m\times n}$
- Transpose of conjugate $(A^\theta) = [\bar{a}_{ji}]_{n\times m}$
- Determinant of matrix | A | (for square matrix A)
  $= \sum_{k=1}^{n} a_{ik}C_{ik}$ where $\{1 \le i \le m\}$
- Trace of matrix $T_r(A) = \sum_{i=1}^{n} a_{ii}$

### SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

- Let the equations be

$a_{11}x + a_{12}y + a_{13}z = b_1$

$a_{21}x + a_{22}y + a_{23}z = b_2$

$a_{31}x + a_{32}y + a_{33}z = b_3$

- These equations can be written in matrix form as

AX = B

$\Rightarrow$ $X = A^{-1}B$

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1}$ , $A = [a_{ij}]_{3\times 3}$ , $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times 1}$

### KINDS OF MATRICES DEPENDING UPON OPERATIONS

- Singular matrix : Square matrix having |A| = 0
- Non singular matrix : Square matrix having $|A| \ne 0$.
- Symmetric : If $a_{ij} = a_{ji}$
- Skew symmetric : if $a_{ij} = -a_{ji}$
- Hermitian : If $a_{ij} = \bar{a}_{ji}$
- Skew hermitian : If $a_{ij} = -\bar{a}_{ji}$
- Orthogonal : Square matrix for which $A^T A = I$
- Unitary : Square matrix for which $A^\theta A = I$
- Idempotent : If $A^2 = A$
- Periodic : If $A^{k+1} = A$ , k is positive integer known as period of A.
- Nilpotent : If $A^\lambda = 0$, $\lambda$ is positive integer known as index of A.