MECHANICAL PROPERTIES OF SOLIDS

1 ELASTICITY

In our study of mechanics so far, we have considered the objects as rigid, i.e., whatever force we apply on objects, they remain undeformed. In reality, when an external force is applied to a body, the body gets deformed in shape or size or both. Also if we remove the external force the deformed body tends to regain its original shape and size. The property by virtue of which a deformed body tends to regain its original shape and size after removal of deforming force is called elasticity. Let us consider two possibilities:

(i) *If the body regains its original size and shape it is said to be perfectly elastic.*

(ii) *If the body does not have any tendency to regain its original shape or size once deforming forces* are removed it is said to be perfectly plastic.

In nature, actual behaviour of bodies lies between these two extreme limits.

1.1 STRESS

When external forces are applied, the body is distorted. i.e. the different portions of the body move relative to each other. Due to these displacements atomic forces (called restoring forces) are set up inside the body to restore the original form. The restoring force per unit area set up inside the body is called stress. This is measured by the magnitude of the deforming force acting per unit area of the body when equilibrium is established. If F is the force applied to an area of cross-section A, then

$$\text{stress} = \frac{F}{A}. \quad \ldots (1)$$

The unit of stress in S.I system is N/m^2 when the stress is normal to the surface, this is called **normal stress**. The normal stress produces a change in length or a change in volume of a body. The normal stress to a wire or body may be compressive or tensile (expansive) according as it produces a decrease or increase in length of a wire or volume of a body. When the stress is tangential to a surface, it is called **tangential (shearing) stress.**

1.3 HOOKE'S LAW AND MODULI OF ELASTICITY

According to Hooke's law, "within elastic limits stress is proportional to strain".

$$\text{i.e.} \quad \frac{\text{stress}}{\text{strain}} = \text{constant} = \lambda \quad \ldots (3)$$

where λ is called modulus of elasticity. Depending upon different types of strain the following three moduli of elasticity are possible.

(i) Young's modulus: When a wire or rod is stretched by a longitudinal force the ratio of the longitudinal stress to the longitudinal strain within the elastic limits is called Young's modulus.

$$\text{Young's modulus } (Y) = \frac{\text{Longitudinal stress}}{\text{Linear strain}}$$

Consider a wire or rod of length L and radius r under the action of a stretching force F applied normal to its faces. Suppose the wire suffers a change in length l then

$$\text{Longitudinal stress} = \frac{F}{\pi r^2}$$

$$\text{Linear strain} = \frac{l}{L}$$

$$\therefore \text{Young's modulus } (Y) = \frac{\frac{F}{\pi r^2}}{\frac{l}{L}}$$

$$Y = \frac{FL}{\pi r^2 l} \quad \ldots (4)$$

(ii) Bulk modulus: When a solid or fluid (liquid or gas) is subjected to a uniform pressure all over the surface, the shape remains the same, but there is a change in the volume. The force per unit area applied normally and uniformly over the surface is called normal stress. The change in volume per unit volume is called volume or bulk strain. The ratio

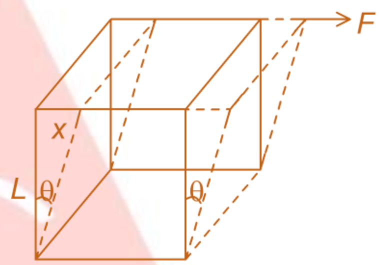
$\dfrac{\text{Volume stress or normal stress}}{\text{Volume strain}}$ is called bulk modulus (*B*).

In symbols $B = -\dfrac{\frac{F}{A}}{\frac{\Delta V}{V}} = -\dfrac{F.V}{A\,\Delta A}$ … (**5**)

The reciprocal of bulk modulus is called compressibility.

$\therefore$ compressibility = $\dfrac{1}{\text{bulk modulus}}$ … (**6**)

(iii) Modulus of Rigidity: According to definition, the ratio of shearing stress to shearing strain is called modulus of rigidity (η). In this case the shape of the body changes but its volume remains unchanged. Consider the case of a cube fixed at its lower face and acted upon by a tangential force *F* on its upper surface of area A as shown in figure.



$\therefore$ shearing stress $= \dfrac{F}{A}$

shearing strain $= \theta = \left(\dfrac{x}{L}\right)$

$\therefore$ $\eta = \dfrac{F}{A\theta} = \dfrac{FL}{Ax}$ … (**7**)

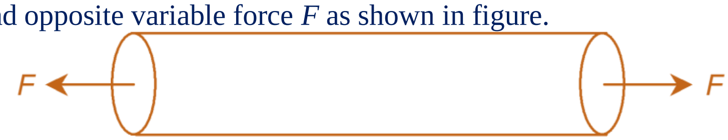
Poisson's Ratio: This is the name given to the ratio of lateral strain to the longitudinal strain. For example, consider a force *F* applied along the length of the wire which elongates the wire along the length while it contracts radially. Then the longitudinal strain = $\dfrac{\Delta L}{L}$ and Lateral strain = $\dfrac{\Delta r}{r}$, where *r* is the original radius and Δr is the change in radius.

$\therefore$ Poisson's ratio (σ) = $-\dfrac{\frac{\Delta r}{r}}{\frac{\Delta L}{L}}$. … (**8**)

The negative sign indicates that change in length and radius are of opposite sign. The theoretical value of σ lies between 1 and 0.5 while the practical value of σ lies between 0 and 0.5.

1.4 STRESS-STRAIN RELATIONSHIP FOR A WIRE SUBJECTED TO LONGITUDINAL STRESS

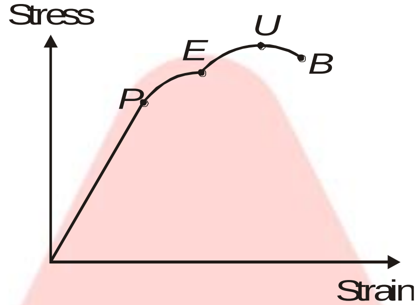
Consider a long wire (made of steel) of cross sectional area *A* and original length *L* in equilibrium under the action of two equal and opposite variable force *F* as shown in figure.



Due to the applied forces the length gets changed to $L + l$.

Then, longitudinal stress = $\dfrac{F}{A}$ and longitudinal strain $= \frac{l}{L}$

The graph between the stress and strain, as the elongation gradually increases is shown in figure. Initially for small value of deformation stress is proportional to strain upto point *P* called as **proportional limit**. Till this point Hook's law is valid and Young's modulus is defined. If deformation further increases, the curve becomes non-linear but still it has elastic property till point *E*. This point *E* is called as **elastic limit**. After elastic limit yielding of wire starts taking place and body starts gaining some permanent deformation.



That is on the removal of force body remains deformed. If deformation is further increased, the wire breaks at point *B* called as breaking point. The stress corresponding to this point is called as breaking stress. Before reaching breaking point the curve has a point where tangent to this curve has zero slope. This point corresponds to maximum stress that the body can sustain. This point is called as ultimate point and stress at this point is called as ultimate strength of wire.

1.5 ELASTIC ENERGY STORED IN A DEFORMED BODY

The elastic energy is measured in terms of work done in straining the body within its elastic limit.

Let *F* be the force applied across the cross-section *A* of a wire of length *L*. Let *l* be the increase in length. Then

$$Y = \frac{\frac{F}{A}}{\frac{l}{L}} = \frac{FL}{Al} \quad \text{or} \quad F = \frac{YAl}{L}$$

If the wire is stretched through a further distance *d l* the work done

$$= F \times d\,l = \frac{YAl}{L}\,dl$$

Total work done in stretching the wire from original length *L* to a length *L* + *l*
(i.e. from *l* = 0 to *l* = *l*)

$$W = \int_0^l \frac{YA\cdot l}{L}\cdot dl$$

$$= \frac{YA}{L}\cdot\frac{l^2}{2} = \frac{1}{2}(AL)\left(\frac{Yl}{L}\right)\left(\frac{l}{L}\right)$$

$$\Rightarrow W = \frac{1}{2} \times \text{volume} \times \text{stress} \times \text{strain} \qquad \ldots \textbf{(9)}$$

2 FLUID STATICS

Fluid statics is the branch of mechanics, which deals with the forces on fluids (liquids and gases) at rest. As far back as 250 B.C. Archimedes, the famous Greek philosopher stated in his works that liquids exert an upward buoyant force on solids immersed in them causing an apparent reduction in their weights. It was about the end of 17th century that Pascal, a French scientist explained the fundamental principles of the subject in a clear manner. The consequences of Pascal's work are far-reaching for it forms the basis of several practical appliances like the fluid pumps, hydraulic presses and brakes, pneumatic drills, etc.

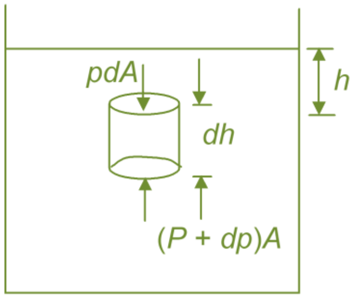
2.1 THRUST AND PRESSURE

A **perfect fluid** resists forces normal to its surface and offers no resistance to forces acting tangential to its surface. A heavy log of wood can be drawn along the surface of water with very little effort because the force applied on the log of wood is horizontal and parallel to the water surface. Thus a fluid is capable of exerting normal stresses on a surface with which it is in contact.

Force exerted perpendicular to a surface is called thrust and thrust per unit area is called pressure.

2.2 VARIATION OF PRESSURE WITH HEIGHT

In all fluids at rest the pressure is a function of vertical dimension. To determine this consider the forces acting on a vertical column of fluid of cross sectional-area dA as shown in figure. The positive direction of vertical measurement h is taken downward. The pressure on the upper side is P, and that on the lower face is $P + dP$. The weight of the element is $\rho gdhdA$. The normal forces on the vertical surfaces of the column do not affect the balance of forces in the vertical direction. Equilibrium of the fluid element in the vertical direction requires.



$$pdA + \rho gdAdh - (p + dp)\, dA = 0$$

$$\Rightarrow \quad dP = \rho gdh$$

This differential relation shows that the pressure in a fluid increases with depth or decreases with increased elevation. Above equation holds for both liquids and gases and agrees with our common observations of air and water pressure.

Liquids are generally treated as incompressible and we may consider their density ρ constant for every part of the liquid. With ρ as constant equation may be integrated as it stands, and the result is

$$P = P_0 + \rho gh \qquad \ldots (\mathbf{10})$$

The pressure P_0 is the pressure at the surface of the liquid where $h = 0$.

2.3 CHARACTERISTICS OF FLUID PRESSURE

(i) Pressure at a point acts equally in all directions.

(ii) Liquids at rest exerts lateral pressure, which increases with depth.

(iii) Pressure acts normally on any area in whatever orientation the area may be held.

(iv) Free surface of a liquid at rest remains horizontal.

(v) Pressure at every point in the same horizontal line is the same inside a liquid at rest.

(vi) Liquid at rest stands at the same height in communicating vessels.

2.4 FORCE DUE TO FLUID ON A PLANE SUBMERGED SURFACE

A surface submerged in liquid such as bottom of a tank or wall of a tank or gate valve in a dam, is subjected to pressure acting normal to its surface and distributed over its area. In problems where the resultant forces are appreciable, we must determine the resultant force due to distribution of pressure on the surface and the position at which resultant force acts.

For system, which is open to earth's atmosphere, the atmospheric pressure P_0 acts over all surfaces and hence, yields a zero resultant. In such cases, then we need to consider only fluid pressure $P = \rho gh$ which is increment above atmospheric pressure.

In general pressure at different point on the submerged surface varies so to calculate resultant force, we divide the surface into number of elementary areas and we calculate force on it first by treating pressure as constant then we integrate it to get net force.

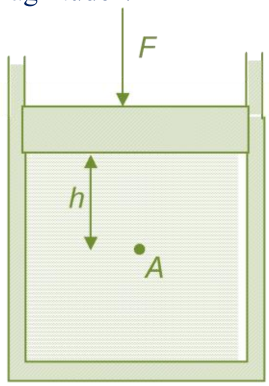
i.e., $F_R = \int PdA$

The point of application of resultant force must be such that the moment of the resultant force about any axis is equal to the moment of the distributed force about the axis.

2.5 PASCAL'S LAW

When we squeeze a tube of toothpaste, the toothpaste comes out of the tube from its opening. This is an example of Pascal's law. When we squeeze the tube, pressure is applied on the tube and this pressure is transmitted everywhere in the tube and forces the toothpaste out of the tube. According to Pascal, "Pressure applied to a fluid confined in a vassel is transmitted to the entire fluid without being diminished in magnitude".

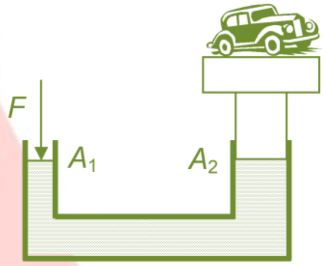
To prove Pascal's law consider a fluid having density ρ in a cylinder fitted with a movable piston. A point A is at the depth h in the fluid. If no force is applied to the piston pressure just below the piston will have same value say P_0. So according to previous discussion pressure at point A will be $P_1 = P_0 + \rho gh$. Now if a force F is applied on the piston, pressure just below the piston will have its value initial pressure plus the increased pressure say $P_0 + P_{ext}$. So new pressure at A can be written as $P_2 = P_0 + P_{ext} + \rho gh$. Therefore increase in the pressure at A will be $P_2 - P_1 = P_{ext}$. That is the increased pressure at the top gets transmitted to A also.



Pascal's law has many important applications and one of the interesting application is in Hydraulic lever. Hydraulic lever is used to lift heavy bodies like automobile by using less effort. It is made of two interconnected vertical cylinders of different cross sectional area A_1 and A_2 and a force F is applied to smaller piston causes much greater force on the larger piston, which can lift the body.

The pressure increased below the smaller piston $= \frac{F}{A_1}$

The same increment of pressure will take place below the larger piston also.



So force transmitted to larger piston $= A_2 \frac{F}{A_1}$.

∴ If the body lifts due to application of F

$$A_2 \frac{F}{A_1} = Mg$$

$$\therefore \quad F = Mg\frac{A_1}{A_2}$$

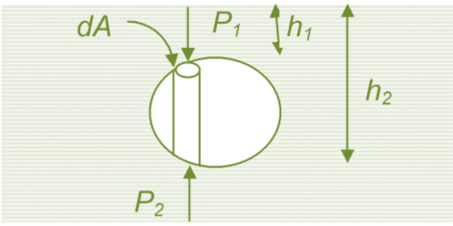
The ratio $\frac{A_1}{A_2}$ is smaller than 1 and thus applied force can be much smaller than the weight Mg that is lifted.

2.6 BUOYANCY AND ARCHIMEDE'S PRINCIPLE

If an object is immersed in or floating on the surface of, a liquid, it experiences a net vertically upward force due to liquid pressure. This force is called as Buoyant force or force of Buoyancy and it acts from the center of gravity of the displaced liquid. According to Archimede's the magnitude of force of buoyancy is equal to the weight of the displaced liquid.

To prove Archimede's principle consider a body totally immersed in a liquid as shown in the figure.

The vertical force on the body due to liquid pressure may be found most easily by considering a cylindrical volume similar to the one shown in figure.



The net vertical force an the element is

$$dF = (P_2 - P_1)\, dA$$

$$= \left[\left(P_0 + \rho\int gh_2\right) - \left(P_0 + \rho\int gh_1\right)\right] dA$$

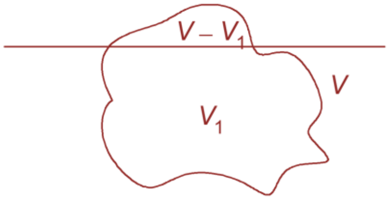
$$= \rho g\int (h_2 - h_1)\, dA$$

but $(h_2 - h_1)\, dA = dV$, the volume of element. Thus

Net vertical force on the body $F = \int \rho g dV = V\rho g$

∴ Force of Buoyancy = $V\rho g$

2.7 EXPRESSION FOR THE IMMERSED VOLUME OF FLOATING BODY



Let a solid of volume V and density ρ floats in a liquid of density σ with a volume V_1 immersed inside the liquid.

The weight of the floating body = $V\rho g$

The weight of the displaced liquid = $V_1\sigma g$

For the equilibrium of the floating body

$$V_1\sigma g = V\rho g$$

or $$\frac{V_1}{V} = \frac{\rho}{\sigma}$$

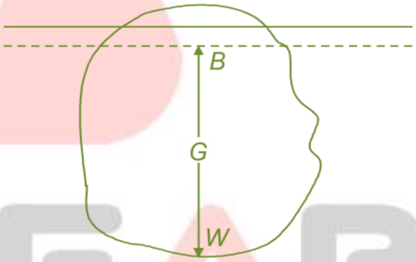
or $$V_1 = \frac{\rho}{\sigma}\times V$$

$$\text{Immersed volume} = \frac{\text{density of solid}}{\text{density of liquid}} \times \text{volume of solid}$$

$$= \frac{\textbf{mass of solid}}{\textbf{density of liquid}}$$

From the above relations it is clear that the density of the solid must be less than the density of the liquid to enable it to float freely in the liquid. However a metal vessel may float in water though the density of metal is much higher than that of water. The reason in that the floating bodies are hollow inside and hence they have a large displaced volume. When they float in water the weight of the displaced water is equal to the weight of the body.

2.8 LAWS OF FLOATION



The principle of Archimedes may be applied to floating bodies to give the laws of floatation as follows:

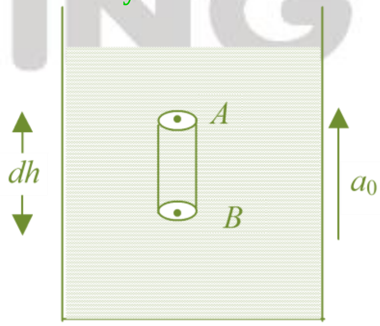
(i) When a body floats freely in a liquid the weight of the body is equal to the weight of the liquid displaced.

(ii)The centre of gravity of the displaced liquid (called the centre of buoyancy) lies vertically above or below the centre of gravity of the body.

2.9 LIQUID IN ACCELERATED VESSEL

A Liquid in accelerated vessel can be considered as in the rigid body motion i.e. motion without deformation as though it were a solid body. As in case of static liquid to determine the pressure variation we apply Newton's law, the same is applicable in case of liquid in accelerated vessel also.

2.9.1 Variation of pressure and force of buoyancy in a liquid kept in vertically accelerated vessel



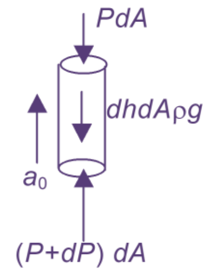
Consider a liquid of density ρ kept in a vessel moving with acceleration a_0 in upward direction.

Let A and B are two points separated vertically by a distance dh. The forces acting an a vertical liquid column of cross sectional area dA are shown in the figure. For the vertical motion of this liquid column,

$$(P + dP)\, dA - PdA - (dh)dA\rho g = (dh)dA\rho a_0$$

$$\Rightarrow dP = \rho\,(g + a_0)\, dh$$

If pressure at the free surface of liquid is P_0 them pressure P at a depth h from the free surface is given by



$$\int_{P_0}^{P} dP = \int_0^h \rho(g + a_0)\, dh$$

$$\Rightarrow P = P_0 + \rho\,(g + a_0)\,h$$

On the basis of similar calculation, pressure at any depth h from free surface in case of liquid in downward accelerated vessel can be written of

$P = P_0 + \rho\,(g - a_0)\,h$

We can generalize the above results and conclude that in liquid for a vertically accelerated vessel the pressure at any depth h below the free surface,

$P = P_0 + \rho g_{eff}\, h$

Where $g_{eff} = g + a_0$ in case of upward acceleration and

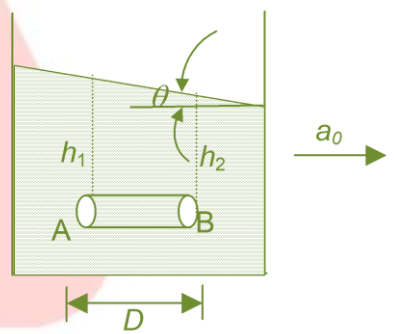
$g_{eff} = g - a_0$ in case of downward acceleration

Also we can say the force of buoyancy F_B on a body in liquid in vertically accelerated vessel is given by $F_B = V\rho g_{eff}$

Where V is volume of liquid displaced by the body.

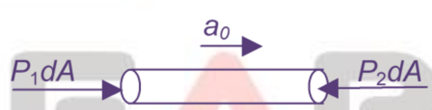
2.9.2 Shape of free surface of liquid in horizontally accelerated vessel

When a vessel filled with liquid accelerates horizontally. We observe its free surface inclined at some angle with horizontal. To find angle θ made by free surface with horizontal consider a horizontal liquid column including two points, A and B at depths h_1 and h_2 from the inclined free surface of liquid as shown in figure. D is length of liquid column. Horizontal forces acting on the liquid column are as shown in diagram, for horizontal motion of column we can write,



$P_1 dA - P_2\, dA = \rho(dA)\, D(a_0)$

$(h_1 - h_2)\, g = D\, a_0$



$$\Rightarrow \frac{h_1 - h_2}{D} = \frac{a_0}{g} \Rightarrow \tan\theta = \frac{a_0}{g} \Rightarrow \theta = \tan^{-1}\frac{a_0}{g}$$

3 FLUID DYNAMICS

3.1 RATE OF FLOW

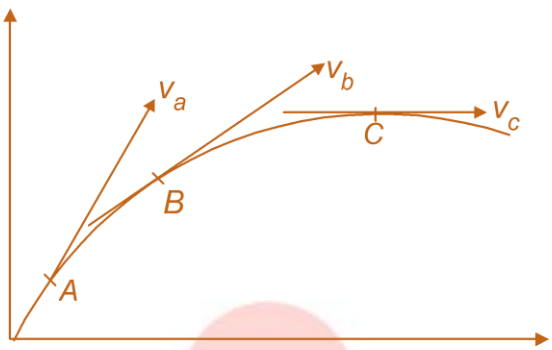
The rate of flow of a liquid is expressed in terms of its volume that flows out from an orifice of known cross-sectional area in unit time. All liquids are practically incompressible and the rate of flow through any section is the same. The rate of flow of a liquid hence is defined as the volume of it that flows across any section in unit time. For example, if the velocity of flow of a liquid is v in a direction perpendicular to two sections A and B of area A and distant l apart and if t be the time taken by the liquid to flow from A to B, we have $vt = l$. Obviously the volume of liquid flowing through section AB is equal to cylindrical column $AB = lA$ or vtA.

$$\therefore \quad \text{Rate of flow of liquid} = \frac{vtA}{t} = vA$$

Sometimes the rate of flow of liquid is expressed in terms of mass of liquid flowing across any section in unit time $vA\rho$ where ρ is the density of liquid.

3.2 STREAMLINES

In steady flow the velocity v at a given point does not change with time. Consider the path ABC of a fluid particle in a steady flow and let the velocities be v_a, v_b, v_c at A, B and C respectively as shown in the figure.

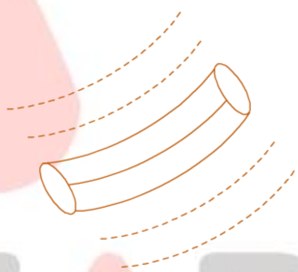


Because v_a does not change with time any particle passing through A will have the same velocity v_a and follow the same path ABC. This curve is called a streamline. A streamline is parallel to the velocity of fluid particle at every point. No two streamlines will ever cross each other. Otherwise any particle at the point of intersection will have two paths to go so that the flow will no more be steady. In non-steady flows the pattern of streamline changes with time. In steady flow if we draw a family of streamlines the tangent to the streamline at any point gives the direction of instantaneous velocity of the fluid at that path of motion. Thus as long as the velocity pattern does not change with time, the streamlines are also the paths along which the particles of the fluid move. In this type of flow the fluid between two surfaces formed by sets of adjacent streamlines is confined to the region between these two surfaces. This is so since the velocity vector has no component perpendicular to any streamline. The flow may be considered to occur in sheets or layers. Hence it is also sometimes called laminar flow. This type of flow is possible only if the velocity is below a certain limiting value called critical velocity. Above this velocity the motion is said to be unsteady or turbulent. The streamlines may be straight or curved depending on to the lateral pressure as it is the same throughout or different.

3.3 TUBE OF FLOW

In a fluid in steady flow, if we select a finite number of streamlines to form a bundle like the streamline pattern shown in figure below the tubular region is called a tube of flow.

The tube of flow is bounded by streamlines so that no fluid can flow across the boundaries of the tube of flow and any fluid that enters at one end must leave at the other end.



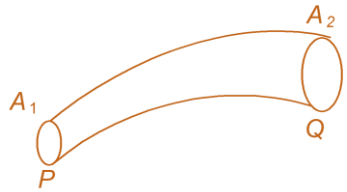
3.4 EQUATION OF CONTINUITY

Consider a thin tube of flow as in the figure. Let A_1 and A_2 be the area of cross-section of the tube perpendicular to the streamlines at P and Q respectively. If v_1 is the velocity of fluid particle at P and v_2 the velocity at Q, then the mass of liquid entering A_1 in a small interval of time Δt is given by $\Delta m_1 = \rho_1 A_1 v_1 \Delta t$.

The mass flux i.e., the mass of fluid flowing per unit time is

$$\frac{\Delta m_1}{\Delta t} = \rho_1 A_1 v_1$$

Let $\Delta t \to 0$ so that v and A does not vary appreciably. The mass flux at P is given by $\rho_1 A_1 v_1$. Similarly the mass flux at $Q = \rho_2 A_2 v_2$ where ρ_1 and ρ_2 are the densities of the fluid at P and Q respectively. Since no fluid crosses the wall of the tube of flow, the mass of fluid crossing each cross-section of the tube per unit time must be the same



$\therefore \quad \rho_1 A_1 v_1 = \rho_2 A_2 v_2$

or $\quad \rho A v = \text{constant}$

This equation expresses the law of conservation of mass in fluid dynamics and is called the **Equation of Continuity**. If the fluid is incompressible as in most liquids ρ = constant

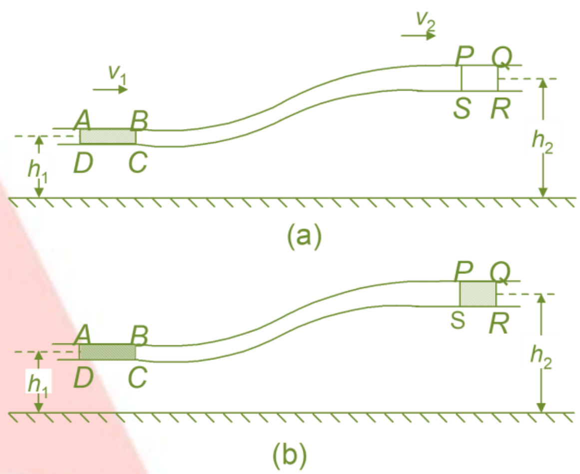
$\therefore \quad A_1v_1 = A_2v_2$ or Av = constant. … (**11**)

The product Av represents the rate of flow of fluid. Hence, v is inversely proportional to the area of cross-section along a tube of flow. Therefore, widely spaced streamlines indicate regions of low speed and closely spaced streamlines indicate regions of high speed.

3.5 BERNOULLI'S EQUATION

It is a fundamental equation in fluid mechanics and can be derived from the work-energy theorem. The theorem states that the work done by all the forces acting on a system is equal to the change in kinetic energy of the system.

Consider a steady, irrotational, non-viscous, incompressible flow of a fluid through a pipe or tube of flow as shown in the figure. A portion of the fluid flowing through a section of pipe line from the position shown in figure (a) to that shown in figure (b).



The pipe line is horizontal at A and P and are at heights h_1 and h_2 respectively from a reference plane. The pipe line gradually widens from left to right. Let P_1, A_1, v_1 and P_2, A_2, v_2 be the pressures, areas of cross-section and velocities at A and P respectively.

Suppose the fluid which escapes the volume $ABCD = A_1\Delta l_1$ flows out at P after some time and occupies $PQRS$ which is equal to $A_2\Delta l_2$ as indicated in (a) and figure (b) respectively. The forces acting on the fluid are the pressure forces P_1A_1 and P_2A_2, the force of gravity at A and P. The work done on the system by the above forces is as follows:

(i) The work done on the system by pressure force P_1A_1 which is $P_1A_1\Delta l_1$

(ii) The work done on the system by pressure force P_2A_2 that is $-P_2A_2\Delta l_2$, the negative sign implying that positive work is done by the system.

(iii) The gravitational work done by the system in lifting the mass m inside from the height h_1 to height $h_2$is given by $mg\,(h_2 - h_1)$ or the work done on the system will be $-\,mg(h_2 - h_1)$

Hence the work done by all the forces on the system is the sum of all the above terms.

$$W = P_1A_1\Delta l_1 - P_2A_2\Delta l_2 - mg\,(h_2 - h_1)$$

Since $A_1\Delta l_1 = A_2\Delta l_2$ we can put $A_1\Delta l_1 = A_2\Delta l_2 = \frac{m}{\rho}$ where ρ is the density of the fluid

$$\therefore\ W = (P_1 - P_2)\,\frac{m}{\rho} - mg\,(h_2 - h_1)$$

The change in kinetic energy of the fluid is $\Delta K = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

From the work-energy theorem we can put the work done on the system is equal to the change in kinetic energy of the system.

$$(P_1 - P_2)\frac{m}{\rho} - mg(h_2 - h_1) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

On multiplying both sides of equation by ρ/m and rearranging we get

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

Since the subscripts 1 and 2 refer to any two locations on the pipeline, we can write in general

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant} \qquad \ldots (\mathbf{12})$$

The above equation is called **Bernoulli's equation** for steady non-viscous incompressible flow. Dividing the above equation by ρg we can rewrite the above as

$$h+\frac{v^2}{2g}+\frac{P}{\rho g} = \text{constant, which is called total head.}$$

Out of the three components of the total head on the left hand side of the above equation the first term '*h*' is called elevation head or gravitational head, the second term $\frac{'v^2'}{2g}$ is called velocity head and the third term $\frac{'P'}{\rho g}$ is called pressure head.

Now if the liquid flows horizontally its potential energy remains constant so that the Bernoulli's equation becomes $P+\frac{1}{2}\rho v^2$ = constant. Further if the fluid is at rest or v = 0, P + ρgh = constant.

The pressure (*P* + ρ*gh*) which would be present also when there is flow is called the static pressure while the term $\frac{1}{2}\rho v^2$ is called the dynamic pressure. The equation shows that for an ideal liquid the velocity increases when pressure decreases and vice-versa.

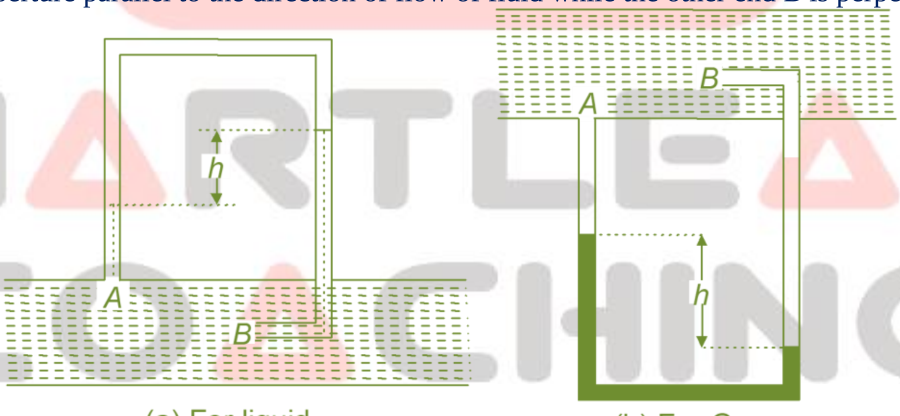
3.6 APPLICATIONS OF BERNOULLI'S THEOREM

We shall consider a few examples wherein Bernoulli's principle is employed.

Filter pump: In a filter pump water flows suddenly from a wide tube into a narrow one. The velocity is increased and pressure reduced far below the atmospheric pressure. The receiver to be exhausted is connected by a side tube with this region of low pressure. Air is carried along with water and in a short time air in the receiver is reduced to a pressure, which is slightly greater than the vapour pressure of water.

Atomizer or sprayer: In flowing out of a narrow tube into the atmosphere or a wider tube, air produces a suction effect. In a sprayer a strong jet of air on issuing from a nozzle of a tube lowers the pressure over another tube, which dips in a liquid. The liquid is sucked up and mixed with air stream and thus a fine mist is produced.

Pitot tube: This device is used to measure the flow speed of liquids and gases in pipes. It consists of a manometer tube connected into pipeline as shown in figure. One of the ends of the manometer tube *A* is connected with its plane of aperture parallel to the direction of flow of fluid while the other end *B* is perpendicular to it.



(a) For liquid (b) For Gases

The pressure and velocity on the left arm of the manometer opening *A* remains the same as they are elsewhere and is equal to static pressure P_a. Since the velocity of fluid at *B* is reduced to zero the pressure is the full arm pressure. Applying Bernoulli's principle, $P_a+\frac{1}{2}\rho v^2 = P_b$ which shows $P_b > P_a$. If *h* is the difference in height of liquid in the manometer and ρ′ the density of manometric liquid,

$$P_a + \rho' gh = P_b$$

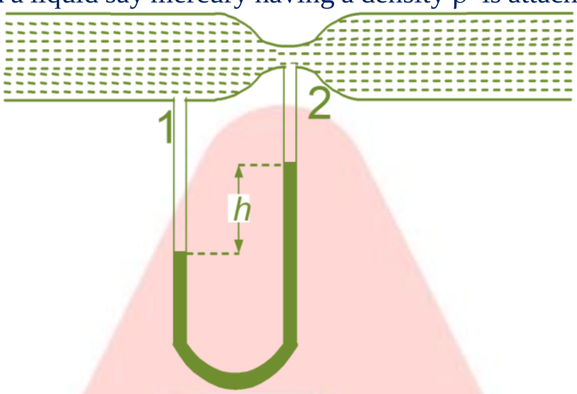
$$\therefore \quad \frac{1}{2}\rho v^2 = \rho' gh$$

In case of Pitot tube ρ = ρ′

$$\therefore \quad v^2 = 2gh$$

In case of gas Pitot tube $v=\sqrt{\dfrac{2gh\rho'}{\rho}}$ … (**13**)

Venturimeter: This is a device based on Bernoulli's principle used for measuring the flow of a liquid in pipes. A liquid of density ρ flows through a pipe of cross sectional area *A*. At the constricted part the cross sectional area is *a*. A manometer tube with a liquid say mercury having a density ρ′ is attached to the tube as shown in figure.



If P_1 is the pressure at point 1 and P_2 the pressure at point 2 we have

$\dfrac{P_1}{\rho}+\dfrac{1}{2}v_1^2=\dfrac{P_2}{\rho}+\dfrac{1}{2}v_2^2$ where v_1 and v_2 are the velocities at these points respectively.

We have $Av_1 = av_2$

$$\therefore \quad v_1=\frac{av_2}{A}$$

$$\therefore \quad \frac{1}{2}v_1^2-\frac{1}{2}v_2^2=\frac{P_2}{\rho}-\frac{P_1}{\rho}=\frac{P_2-P_1}{\rho}$$

$$\therefore \quad v_1^2-\frac{A^2v_1^2}{a^2}=\frac{2(P_2-P_1)}{\rho},$$

$$v_1^2\left(1-\frac{A^2}{a^2}\right)=\frac{2(P_2-P_1)}{\rho}$$

$$v_1^2=\frac{2(P_2-P_1)}{\left(1-\frac{A^2}{a^2}\right)\rho}=\frac{2a^2(P_2-P_1)}{(a^2-A^2)\rho}$$

If $a < A$,

$$v_1^2=\frac{2a^2(P_1-P_2)}{(A^2-a^2)\rho}$$

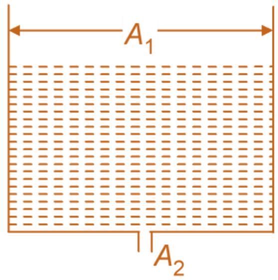
$$v_1=\sqrt{\frac{2(P_1-P_2)a^2}{(A^2-a^2)\rho}}$$

Volume of liquid flowing through the pipe per sec. = $Av_1 = Q$

$$Q=\boldsymbol{Aa}\sqrt{\frac{\boldsymbol{2(P_1-P_2)}}{\boldsymbol{(A^2-a^2)\rho}}} \quad \text{… (14)}$$

Speed of Efflux

As shown in figure a tank of cross-sectional area A_1, filled to a depth h with a liquid of density ρ. There is a hole of cross-sectional area A_2 at the bottom and the liquid flows out of the tank through the hole $A_2 << A_1$



Let v_1 and v_2 be the speeds of the liquid at A_1 and A_2.

As both the cross-sections are open to the atmosphere, the pressures there equals the atmospheric pressure P_0. If the height of the free surface above the hole is h,

Bernoulli's equation gives

$$P_0 + \frac{1}{2}\rho v_1^2 + \rho gh = P_0 + \frac{1}{2}\rho v_2^2$$

By the equation of continuity,

$$A_1 v_1 = A_2 v_2$$

$$P_0 + \frac{1}{2}\rho\left[\frac{A_2}{A_1}\right]^2 v_2^2 + \rho gh = P_0 + \frac{1}{2}\rho v_2^2$$

$$\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] v_2^2 = 2gh$$

$$\Rightarrow v_2 = \sqrt{\frac{2gh}{1-\left(\frac{A_2}{A_1}\right)^2}} \quad \ldots (15)$$

If $A_2 <<< A_1$, the equation reduces to $v_2 = \sqrt{2gh}$ … **(16)**

The speed of efflux is the same as the speed a body would acquire in falling freely through a height h. This is known as Torricelli's theorem.

4 SURFACE TENSION

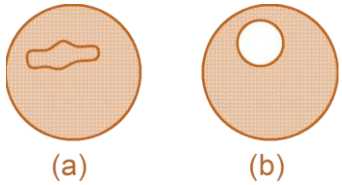
4.1 INTRODUCTION

The surface of a liquid behaves somewhat like a stretched elastic membrane. Just as a stretched elastic membrane has a natural tendency to contract and occupy a minimum area, so also the surface of a liquid has got the natural tendency to contract and occupy a minimum possible area as permitted by the circumstances of the liquid mass. We have many evidences in support of this fact, which we discuss now

(i) We very often find that a small quantity of a liquid spontaneously takes a spherical shape, e.g., rain drops, small quantities of mercury placed on a clean glass plate etc. Now, for a given volume, a sphere has the least surface area. Thus, this fact shows that a liquid always tends to have the least surface area.

(ii) If we immerse a camel-hair brush in water, its hairs spread out, but the moment it is taken out of water, they all cling together as though bound by some sort of elastic thread.

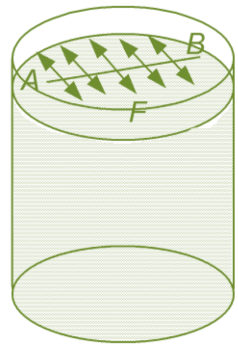
Experimental demonstration: Make a circular loop of wire and dip it in a soap solution. Take it out of the solution and find a thin film of soap solution across the loop. Place a moistened cotton loop gently on the film. The thread will lie on the film in an irregular manner as shown in figure (a).



Now prick the film inside the cotton thread loop by a pin. At once the thread will spontaneously take the shape of a circle as shown in figure (b). The thread has a fixed perimeter. For a given perimeter a circle has got the maximum area. Hence the remaining portion of the film occupies the minimum area.

4.2 DEFINITION

The above evidences and experiment prove beyond doubt that the surface of a liquid behaves like a stretched elastic membrane having a natural tendency to contract and occupy a minimum possible area as permitted by the circumstances of the liquid mass. It is defined as ***"the property of the surface of a liquid by virtue of which it tends to contract and occupy the minimum possible area is called surface tension and is measured by the force per unit length of a line drawn on the liquid surface, acting perpendicular to it and tangentially to the surface of the liquid"***.



Let an imaginary line *AB* on the surface of liquid of length *L* as shown in figure and force on this line is *F*, So surface tension

$$T = \frac{F}{L} \qquad \dots (1)$$

Surface tension has dimensions [MT^{-2}] and Its unit is Newton per metre (Nm^{-1}).

It depends on temperature. The surface tension of all liquids decreases linearly with temperature

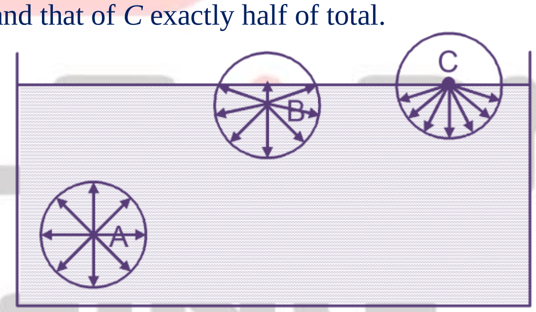
It is a scalar quantity and become zero at a critical temperature.

4.3 MOLECULAR THEORY OF SURFACE TENSION

The surface tension of a liquid arises out of the attraction of its molecules. Molecules of a fluid (liquid and gas) attract one another with a force. It depends on the distance between molecules. The distance up to which the force of attraction between two molecules is appreciable is called the molecular range, and is generally of the order of 10^{-9} metre. A sphere of radius equal to the molecular range drawn around a molecule is called sphere of influence of the molecules lying within the sphere of influence.

Consider three molecules *A*, *B* and *C* having their spheres of influence as shown in the figure. The sphere of influence of *A* is well inside the liquid, that of *B* partly outside and that of *C* exactly half of total.

Molecules like *A* do not experience any resultant force, as they are attracted equally in all directions. Molecules like *B* or *C* will experience a resultant force directed inward. Thus the molecules well inside the liquid will have only kinetic energy but the molecules near the surface will have kinetic energy as well as potential energy which is equal to the work done in placing them near the surface against the force of attraction directed inward.

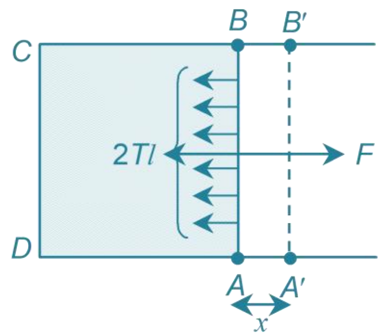


4.4 SURFACE ENERGY

Any strained body possesses potential energy, which is equal to the work done in bringing it to the present state from its initial unstrained state. The surface of a liquid is also a strained system and hence the surface of a liquid also has potential energy, which is equal to the work done in creating the surface. This energy per unit area of the surface is called surface energy.

To derive an expression for surface energy consider a wire frame equipped with a sliding wire *AB* as shown in figure. A film of soap solution is formed across *ABCD* of the frame. The side *AB* is pulled to the left due to surface tension. To keep the wire in position a force F, has to be applied to the right. If *T* is the surface tension and *l* is the length of *AB*, then the force due to surface tension over AB is $2lT$ to the left because the film has two surfaces (upper and lower).

Since the film is in equilibrium, $F = 2lT$



Now, if the wire *AB* is pulled to the right, energy will flow from the agent to the film and this energy is stored as potential energy of the surface created just now. Let the wire be pulled slowly through *x*.

Then the work done = energy added to the film from the agent

$= F.x = 2lTx$

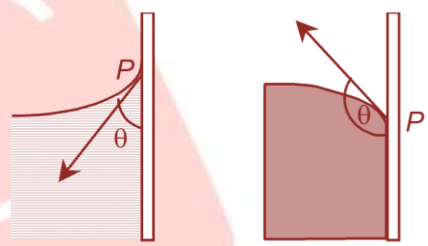
Potential energy per unit area (Surface energy) of the film $= \frac{2lTx}{2lx} = T$ … (**2**)

Thus surface energy of the film is numerically equal to its surface tension.

Its unit is joule per square metre (Jm^{-2}).

4.5 ANGLE OF CONTACT

When a solid body in the form of a tube or plate is immersed in a liquid, the surface of the liquid near the solid is, in general, curved. It is defined as *the angle between the tangents to the liquid surface and the solid surface at the point of contact, for that pair of solid and liquid is called* ***angle of contact***.



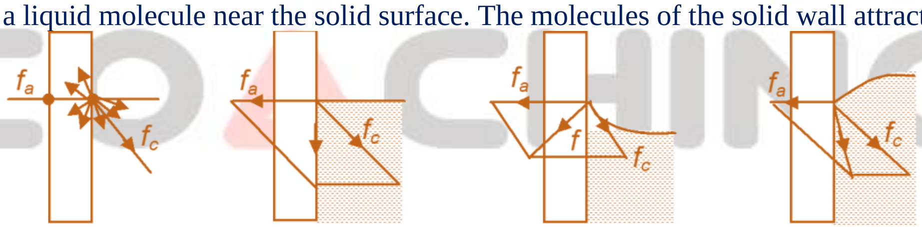
(a) (b)

For example when glass strip is dipped in water and mercury as shown in figure (a) and (b) respectively, the angle θ is angle of contact, which is acute in case of water and obtuse in case of mercury.

4.6 ADHESIVE FORCE AND COHESIVE FORCE

The angle of contact arises due to adhesive and cohesive forces on the molecules of the liquid which lie near the solid surface. Forces of attraction between molecules of different substances are called **adhesive forces** and the forces of attraction between molecules of the same substance are called **cohesive forces**.

Consider a liquid molecule near the solid surface. The molecules of the solid wall attract this molecule.



(a) (b) (c) (d)

These forces are adhesive and are distributed over 180° and hence their resultant acts at right angles to the solid wall and the forces of cohesion, i.e., attraction by liquid molecules, are distributed over 90° and hence their resultant will be inclined at 45° to the solid wall as shown in figure (a) and (b). Let f_a be the resultant adhesive force and f_c be the resultant cohesive force. Then the angle between f_a and f_c is 135°. Let f be their resultant making angle θ with f_a. Then

$$\tan\theta = \frac{f_c \sin 135^\circ}{f_a + f_c \cos 135^\circ} = \frac{f_c}{\sqrt{2} f_a - f_c} \qquad \dots (3)$$

Case I: If $\sqrt{2} f_a = f_c$, then tan θ = ∝ or θ = 90°, i.e., resultant will lie along the solid surface.

Case II: If $\sqrt{2}f_a > f_c$, tanθ is positive and hence θ < 90°, i.e., the resultant lies inside the solid as in figure(c).

Case III: If $\sqrt{2}f_a < f_c$, tanθ is negative and hence θ > 90°, i.e., the resultant lies inside the liquid as shown in figure (d).

A liquid cannot permanently withstand a shearing force, as it has no rigidity. Hence the free surface of a liquid will be at right angles to the resultant force.

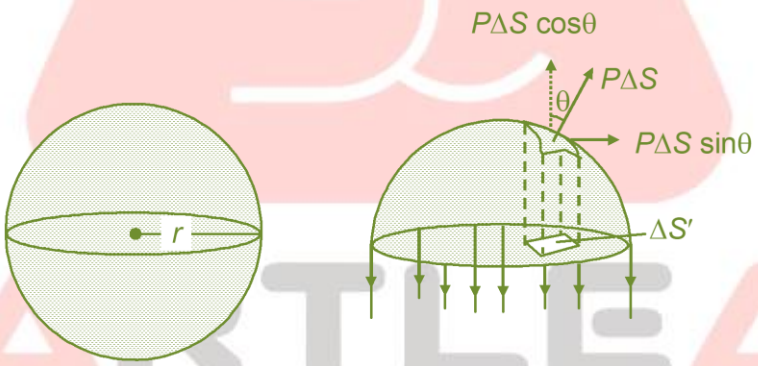
Thus, in the first case, when the resultant force *f* acts along the solid surface, the liquid surface is at right angles to the solid surface.

In the second case when the cohesive force is smaller than the adhesive force such that $f_c < \sqrt{2}f_a$, the resultant is inside the solid and so the liquid surface is inclined to the solid surface at a small angle. The free surface of the liquid is, in consequence, concave upwards near the wall. The concavity of the surface decreases gradually and the free surface becomes horizontal at a large distance from the wall. In this case it is said that the liquid wets the solid, e.g., water wets glass.

In the third case when the cohesive force (f_c) is larger than the adhesive force such that $f_c > \sqrt{2}f_a$ the resultant lies inside the liquid and so the liquid surface is inclined to the solid surface at a large angle. Thus in this case the liquid surface is convex upwards near the wall. In this case it is said that the liquid surface does not wet the solid, e.g., mercury and glass.

4.7 EXCESS PRESSURE

The pressure inside a liquid drop or a soap bubble must be in excess of the pressure outside the bubble drop because without such pressure difference a drop or a bubble cannot be in stable equilibrium. Due to surface tension the drop or bubble has got the tendency to contract and disappear altogether. To balance this, there must be an excess of pressure inside the bubble.



To obtain a relation between the excess of pressure and the surface tension, consider a water drop of radius *r* and surface tension *T*. Divide the drop into two halves by a horizontal plane passing through its center as shown in figure and consider the equilibrium of one-half, say, the upper half. The forces acting on it are:

(i) forces due to surface tension distributed along the circumference of the section.

(ii) outward thrusts on elementary areas of it due to excess pressure.

Obviously, both the types of forces are distributed. The first type of distributed forces combine into a force of magnitude $2\pi r \times T$. To find the resultant of the other type of distributed forces, consider an elementary area ΔS of the surface. The outward thrust on $\Delta S = p\,\Delta S$ where *p* is the excess of the pressure inside the bubble. If this thrust makes an angle θ with the vertical, then it is equivalent to $\Delta Sp\cos\theta$ along the vertical and $\Delta Sp\sin\theta$ along the horizontal. The resolved component $\Delta Sp\sin\theta$ is ineffective as it is perpendicular to the resultant force due to surface tension. The resolved component $\Delta Sp\cos\theta$ contributes to balancing the force due to surface tension.

The resultant outward thrust = $\Sigma\Delta Sp\cos\theta$

$= p\Sigma\Delta S\cos\theta$

$= p\Sigma\Delta S'$ where $\Delta S' = \Delta S\cos\theta$

= area of the projection of ΔS on the horizontal dividing plane

$= p \times \pi r^2 \quad (\because \Sigma\Delta S' = \pi r^2)$

For equilibrium of the bubble we have

$$\pi r^2 p = 2\pi r.T$$

$$\text{or, } p = \frac{2T}{r} \quad \dots (4)$$

If it is a **soap bubble**, the resultant force due to surface tension is $2\pi r.2T$, because a bubble has two surfaces. Hence for the equilibrium of a bubble we have

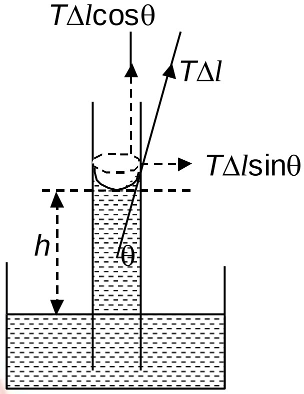
$$\pi r^2 p = 4\pi r T$$

$$\text{or, } p = \frac{4T}{r} \quad \dots (5)$$

4.8 CAPILLARY ACTION

When a glass tube of very fine bore called a capillary tube is dipped in a liquid (like water,) the liquid immediately rises up into it due to the surface tension. This phenomenon of rise of a liquid in a narrow tube is known as capillarity.

Suppose that a capillary tube of radius r is dipped vertically in a liquid. The liquid surface meets the wall of the tube at some inclination θ called the angle of contact. Due to surface tension a force, ΔlT acts on an element Δl of the circle of contact along which the liquid surface meets the solid surface and it is tangential to the liquid surface at inclination θ to the wall of the tube. (The liquid on the wall of the tube exerts this force. By the third law of motion, the tube exerts the same force on the liquid in the opposite direction.) Resolving this latter force along and perpendicular to the wall of the tube, we have $\Delta lT\cos\theta$ along the tube vertically upward and $\Delta lT\sin\theta$ perpendicular to the wall. The latter component is ineffective. It simply compresses the liquid against the wall of the tube. The vertical component $\Delta lT\cos\theta$ pulls the liquid up the tube.



The total vertical upward force = $\Sigma\Delta lT\cos\theta$

$$= T\cos\theta\Sigma\Delta l = T\cos\theta\, 2\pi r$$

$$(\because \Sigma\Delta l = 2\pi r).$$

Due to this upward pull liquid rises up in the capillary tube till it is balanced by the downward gravitational pull. If h is the height of the liquid column in the tube up to the bottom of the meniscus and v is the volume of the liquid above the horizontal plane touching the meniscus at the bottom, the gravitational pull, i.e., weight of the liquid inside the tube is $(\pi r^2 h + v)\rho g$.

For equilibrium of the liquid column in the tube

$$2\pi rT\cos\theta = (\pi r^2 h + v)\rho g.$$

If volume of the liquid in meniscus is negligible then,

$$2\pi rT\cos\theta = (\pi r^2 h)\rho g.$$

$$h = \frac{2T\cos\theta}{r\rho g} \quad \dots (6)$$

The small volume of the liquid above the horizontal plane through the lowest point of the meniscus can be calculated if θ is given or known. For pure water and glass θ = 0° and hence the meniscus is hemispherical.

∴ v = volume of the cylinder of height r – volume of hemisphere.

$$= \pi r^3 - \frac{1}{2}\frac{4\pi}{3}r^3$$

$$= \pi r^3 - \frac{2}{3}\pi r^3 = \frac{1}{3}\pi r^3$$

∴ For water and glass

$$2\pi rT = \left(\pi r^2 h + \frac{\pi r^3}{3}\right)\rho g$$

$$2T = r\left(h + \frac{r}{3}\right)\rho g$$

$$h = \frac{2T}{r\rho g} - \frac{r}{3}$$

For a given liquid and solid at a given place as ρ, T, θ and g are constant,

∴ hr = constant

i.e., lesser the radius of capillary greater will be the rise and vice-versa.

5 VISCOSITY

If water in a tube is whirled and then left to itself, the motion of the water stops after some time. This is a very common observation. What stops the motion? There is no external force to stop it. A natural conclusion is, therefore, that whenever there is relative motion between parts of a fluid, internal forces are set up in the fluid, which oppose the relative motion between the parts in the same way as forces of friction operate when a block of wood is dragged along the ground. This is why to maintain relative motion between layers of a fluid an external force is needed.

"This property of a fluid by virtue of which it oppose the relative motion between its different layers is known as viscosity and the force that is into play is called the viscous force".

Consider the slow and steady flow of a fluid over a fixed horizontal surface. Let v be the velocity of a thin layer of the fluid at a distance x from the fixed solid surface. Then according to Newton, the viscous force acting tangentially to the layer is proportional to the area of the layer and the velocity gradient at the layer. If F is the viscous force on the layer, then

$F \propto A$ where A is the area of the layer

$$\propto -\frac{dv}{dx}$$

The negative sign is put to account for the fact that the viscous force is opposite to the direction of motion.

$$\Rightarrow \quad F = -\eta A \frac{dv}{dx} \quad \dots (7)$$

where η is a constant depending upon the nature of the liquid and is called the coefficient of viscosity and

Velocity gradient = $\frac{dv}{dx}$

If $A = 1$ and $\frac{dv}{dx} = 1$, we have $F = -\eta$

Thus the coefficient of viscosity of a liquid may be defined as *the viscous force per unit area of the layer where velocity gradient is unity.*

The coefficient of viscosity has the dimension $[ML^{-1}T^{-1}]$ and its unit is Newton second per square metre (Nsm^{-2}) or kilograme per metre per second ($kgm^{-1}s^{-1}$). In CGS the unit of viscosity is Poise

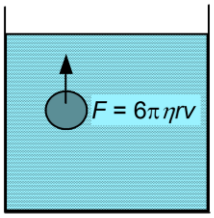
5.1 STOKE'S LAW

When a solid moves through a viscous medium, its motion is opposed by a viscous force depending on the velocity and shape and size of the body. The energy of the body is continuously decreases in overcoming the viscous resistance of the medium. This is why cars, aeroplanes etc. are shaped streamline to minimize the viscous resistance on them.

The viscous drag on a spherical body of radius r, moving with velocity v, in a viscous medium of viscosity η is given by

$$F_{viscous} = 6\pi\eta rv.$$

This relation is called **Stokes' law**



This law can be deduced by the method of dimensions. By experience we guess that the viscous force on a moving spherical body may depend on its velocity, radius and coefficient of viscosity of the medium. We may then write

$$F = k\, v^a r^b \eta^c$$

where k is a constant (dimensionless) and a, b and c are the constants to be determined.

By taking dimensions of both sides, we have

$$MLT^{-2} = (LT^{-1})^a L^b (ML^{-1}T^{-1})^c$$

or, $$MLT^{-2} = M^c L^{a+b-c} T^{-a-c}$$

Equating powers of M, L and T we have

$$c = 1$$

$$a + b - c = 1$$

and $$-a - c = -2$$

Solving we have $a = 1$, $b = 1$ and $c = 1$

$$F = k\eta r v$$

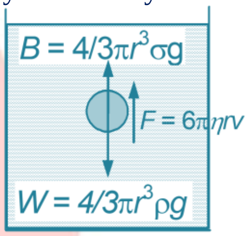
Experimentally k is found to be 6π;

$$F = 6\pi\eta r v. \qquad \ldots \textbf{(15)}$$

5.2 TERMINAL VELOCITY

Let the body be driven by a constant force. In the beginning the viscous drag on the body is small. Velocity is small and so the body is accelerated through the medium by the driving force. With the increase of velocity of the body the viscous drag on it will also increase and eventually when it becomes equal to the driving force, the body will acquire a constant velocity. This velocity is called the terminal velocity of the body.

Consider the downward motion of a spherical body through a viscous medium such as a ball falling through liquid. If r is the radius of the body, ρ the density of the material of the body and σ is the density of the liquid then,



the weight of the body = $\frac{4\pi}{3} r^3 \rho g$,downwards

and the buoyancy of the body = $\frac{4\pi}{3} r^3 \sigma g$,upwards

The net downward driving force = $\frac{4\pi}{3} r^3 (\rho - \sigma) g$

If v is the terminal velocity of the body, then the viscous force on the body is

$$F = 6\pi\eta r v$$

For no acceleration of the body we have

$$6\pi\eta r v = \frac{4\pi}{3} r^3 (\rho - \sigma) g \quad \text{or,} \quad v = \frac{2}{9} \cdot \frac{r^2 g (\rho - \sigma)}{\eta} \qquad \ldots \textbf{(16)}$$

MIND MAP

1. Stress = $\frac{F}{A}$

Normal stress = $\frac{F_n}{A}$

Tangential stress = $\frac{F_t}{A}$

2. Strain

Longitudinal strain = $\frac{\Delta L}{L}$

Volume strain = $\frac{\Delta V}{V}$

$\frac{x}{L}$

3. **Hook's Law:** within elastic limit stress $\propto$ strain

(i) Young's modulus (Y)

$\frac{F}{A} = Y\frac{l}{L}$

(ii) Bulk modulus (B)

$\frac{F}{A} = -B\frac{\Delta V}{V}$

(iii) Modulus of rigidity (η)

$\frac{F}{A} = \eta\frac{x}{L}$

4. Energy stored in a deformed body per unit volume

$= \frac{1}{2} \times$ stress $\times$ strain

1. $T = \frac{F}{L}$

$T = \frac{W}{\Delta A}$

Angle of contact:

(i) Acute: Adhesion > cohesion; Meniscus is concave

(ii) Right angle: Adhesion = cohesion; Meniscus is plane

(iii) Obtuse: Adhesion < cohesion; Meniscus is convex.

FLUID & SOLID

pressure:

(i) Inside a drop

$p = \frac{2T}{r} = p_i - p_o$

(ii) Inside a bubble

$p = \frac{4T}{r} = p_i - p_o$

2. **Capillarity:**

(i) $h = \frac{2T\cos\theta}{R\rho g} = \frac{2T}{r\rho g}$

(ii) hr = constant

Fluid Statics

1. Variation of pressure in a liquid $\rightarrow$ Pressure varies along the depth of a liquid as ρgh.
2. Force of Buoyancy

   = Weight of displaced liquid = $V\rho g$
3. Center of Buoyancy is a point from which force of buoyancy acts. It is center of gravity of displaced liquid.

Fluid Dynamics

1. Continuity equation $A_1V_1 = A_2V_2$
2. Bernoullis theorem $P + \rho gh + \frac{1}{2}\rho v^2$ = constant
3. Velocity of efflux $= \sqrt{\frac{2gh}{1 - \frac{A_2^2}{A_1^2}}} \approx \sqrt{2gh}$

4. **Newton's Law**

$F = -\eta A\frac{dv}{dx}$

5. **Stokes Law:**

(i) $F = 6\pi\eta rv$

(ii) Terminal velocity

$v_T = \frac{2}{9}r^2\frac{(\rho - \sigma)}{\eta}g$