Permutations & Combinations

1 INTRODUCTION

A branch of mathematics where we count number of objects or number of ways of doing a particular job without actually counting them, is known as combinatorics and in this chapter we will deal with elementary combinatorics.

For example, if in a room there are five rows of chairs and each rows contains seven chairs, then without counting them we can say, total number of chairs is 35. We start this chapter with principle of product or fundamental principle of counting.

2 FUNDAMENTAL PRINCIPLE OF COUNTING

2.1 MULTIPLICATION PRINCIPLE OF COUNTING

If a job can be done in m ways, and when it is done in any one of these ways another job can be done in n, then both the jobs together can be done in mn ways. The rule can be extended to move number of jobs.

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2.2 ADDITION PRINCIPLE OF COUNTING

If a job can be done in m ways and another job can be done in n ways then either of these jobs can be done in $m + n$ ways. The rule can be extended to more number of jobs.

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3 PERMUTATIONS

Each of different arrangement which can be made by taking some or all of a number of things is called a ***permutation***. It is assumed that

- the given things let us say there are n of them are all distinct, that is, no two are alike.
- the arrangement is one of placing one thing next to another as in a straight line; hence it is also known as linear permutation.
- In any arrangement any one thing is used only once. In other words, there is no repetition.

3.1 COUNTING FORMULAE FOR PERMUTATION

To find the value of ^nP_r

Suppose there are r blank spaces in a row and n different letters. The number of ways of filling up the blank spaces with n different letters is the number of ways of arranging n things r at a time, i.e. ^nP_r. It must be noted that each space has to be filled up with only one letter.

1	2	3		r

The first space can be filled in n ways. Having filled it, there are $n – 1$ letters left and therefore the second space can be filled in $n – 1$ ways. Hence the first two spaces can be filled in $n(n – 1)$ ways. When the first two spaces are filled, there are $n – 2$ letters left, so that the third space can be filled in $n – 2$ ways. Therefore the first three spaces can be filled in $n(n – 1)(n – 2)$ ways; proceeding like this, the r spaces can be filled in $n(n – 1)(n – 2) \ldots [n – (r – 1)]$ ways.

The number of permutations of n things taken r at a time is denoted as ^nP_r and its value is equal to: $^nP_r = n(n – 1)(n – 2) \ldots\ldots (n – r + 1)$

$$= \frac{n!}{(n-r)!}$$ (using factorial notation $n! = n(n – 1) \ldots$ 3.2.1.) where $0 \le r \le n$.

In particular

- The number of permutations of n different things taken all at a time = ${}^nP_n = n!$
- ${}^nP_0 = 1$, ${}^nP_1 = n$ and ${}^nP_{n-1} = {}^nP_n = n!$
- ${}^nP_r = n\,({}^{n-1}P_{r-1})$ where $r = 1, 2, \ldots\ldots n$

3.2 IMPORTANT RESULTS

Number of permutations under certain conditions:

- Number of permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement is $r\,{}^{n-1}P_{r-1}$.
- Number of permutations of n different things, taken r at a time, when a particular thing is never taken in any arrangement is ${}^{n-1}P_r$.
- Number of permutations of n different things, taken r at a time, when m particular things are never taken in any arrangement is ${}^{n-m}P_r$.
- Number of permutations of n different things, taken all at a time, when m specified things always come together is $(m\,!)\,(n - m + 1)\,!$.
- Number of permutations of n different things, taken all at a time, when m specified things never come together is $n! - m!\,(n - m + 1)\,!$.

3.3 PERMUTATION OF *n* DISTINCT OBJECT WHEN REPETITION IS ALLOWED

- The number of permutations of n different things taken r at time when each thing may be repeated any number of times is n^r.

In other words if a job can be completed in n ways and it is to be repeated r number of times then the total number of ways is n^r.

As an example the number of 5 digit numbers, in which digits are not repeated is $9({}^9p_4)$ while when repetition is allowed is 9×10^4.

Also if n distinct things are arrangement at r places when repetition is allowed, then the number of arrangements are $\underbrace{n \times n \times \ldots\ldots \times n}_{r \text{ times}} = n^r$.

4 CIRCULAR PERMUTATIONS

In the event of the given n things arranged in a circular or even elliptical permutation – and in this case the first and the last thing in the arrangement are indistinguishable – the number of permutations is $(n - 1)\,!$.

For example is 20 persons are circularly arranged, the number of arrangements is 19 !.

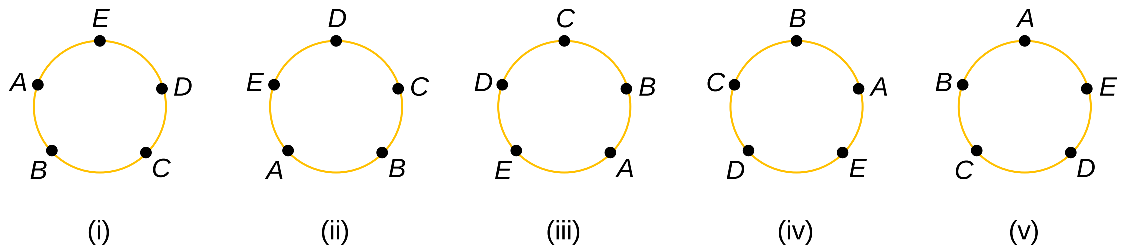
However, in the case of circular permutations wherein clockwise and anticlockwise orders need not be differentiated – as in the case of differently coloured beads or different flowers made into a garland, the number of permutations is $\frac{(n-1)\,!}{2}$ where the number of beads or flowers is taken as n. In details the concept is explained as follows:

4.1 ARRANGEMENTS AROUND A CIRCULAR TABLE

Consider five persons A, B, C, D, E be seated on the circumference of a circular table in order which has no head now, shifting A, B, C, D, E one position in anticlockwise direction we will get arrangements as shown in following figure:

We observe that arrangements in all figures are different.

Thus, the number of circular permutations of n different things taken all at a time is $(n - 1)!$, if clockwise and anticlockwise orders are taken as different.

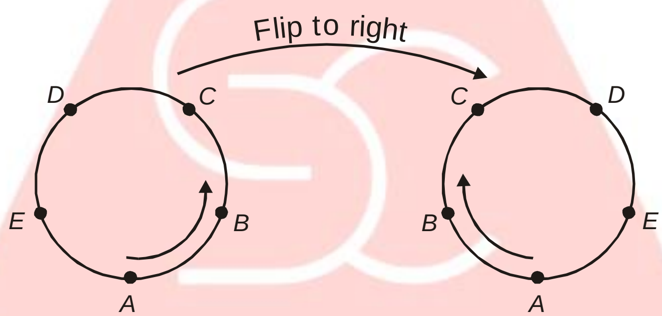


4.2 ARRANGEMENTS OF BEADS OR FLOWERS (ALL DIFFERENT) AROUND A CIRCULAR NECKLACE OR GARLAND

Consider five beads *A*, *B*, *C*, *D*, *E* in a necklace or five flowers *A*, *B*, *C*, *D*, *E* in a garland etc. *If the necklace or garland on the left is turned over we obtain the arrangement on the right, i.e., anticlockwise and clockwise order of arrangement is not different we will get arrangements as follows:*

we see that arrangements in above figures are not different.

Then the number of circular permutations of *n* different things taken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anticlockwise orders are taken as not different.



4.3 NUMBER OF CIRCULAR PERMUTATIONS OF *n* DIFFERENT THINGS TAKEN *r* AT A TIME

CASE I : If clockwise and anticlockwise orders are taken as different, then the required number of circular permutations = $\frac{^nP_r}{r}$.

5 COMBINATIONS

Each of different grouping or selections that can be made by some or all of a number of given things without considering the order in which things are placed in each group, is called ***combinations***.

5.1 COUNTING FORMULAE FOR COMBINATIONS

The number of combinations (selections or groupings) that can be formed from *n* different objects taken *r* at a time is denoted by ^nC_r and its value is equal to

$$^nC_r = \frac{n!}{(n-r)!\ r!} \quad (0 \le r \le n)$$

as $$^nC_r = \frac{^nP_r}{r\,!}$$

as in a permutation the arrangement of *r* selected objects out of *n*, is done in *r*! ways and in combination arrangement in a group is not considered.

In particular

- $^nC_0 = {^nC_n} = 1$ i.e. there is only one way to select none or to select all objects out of *n* distinct objects.
- $^nC_1 = n$ There are *n* ways to select one thing out of *n* distinct things.

- ${}^nC_r = {}^nC_{n-r}$
  Therefore ${}^nC_x = {}^nC_y \Leftrightarrow x = y$ or $x + y = n$.
- If n is odd then the greatest value of ${}^nC_r$ is ${}^nC_{\frac{n+1}{2}}$ or ${}^nC_{\frac{n-1}{2}}$.
- If n is even then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$.

5.2 IMPORTANT RESULTS OF COMBINATIONS (SELECTIONS)

- The number of ways in which r objects can be selected from n distinct objects if a particular object is always included is ${}^{n-1}C_{r-1}$.
- The number of ways in which r objects can be selected from n distinct objects if a particular object is always excluded is ${}^{n-1}C_r$.
- The number of ways in which r objects can be selected from n distinct objects if m particular objects are always included is ${}^{n-m}C_{r-m}$.
- The number of ways in which r objects can be selected from n distinct objects if m particular objects are always excluded is ${}^{n-m}C_r$.

6 SELECTION FROM DISTINCT/IDENTICAL OBJECTS

6.1 SELECTION FROM DISTINCT OBJECTS

- The number of ways (or combinations) of selection from n distinct objects, taken at least one of them is

  $${}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n - 1$$

  Logically it can be explained in two ways, as one can be selected in ${}^nC_1$ ways, two in ${}^nC_2$ ways and so on ……. and by addition principle of counting the total number of ways of doing either of the job is ${}^nC_1 + {}^nC_2 + ...+ {}^nC_n$
  Also, for every object, there are two choices, either selection or non-selection. Hence total choices are 2^n. But this also includes the case when none of them is selected. Therefore the number of selections, when atleast one is selected = $2^n - 1$.

6.2 SELECTION FROM IDENTICAL OBJECTS

1. The number of selections of r ($r \le n$) objects out of n identical objects is 1.
2. The number of ways of selections of at least one object out of n identical object is n.
3. The number of ways of selections of at least one out of $a_1 + a_2$ ……. $+ a_n$ objects, where a_1 are alike of one kind, a_2 are alike of second kind, and so on ……..a_n are alike of n^{th} kind, is
   $(a_1 + 1)(a_2 + 1) ……. (a_n + 1) - 1$.
4. The number of ways of selections of atleast one out of $a_1 + a_2 + a_3 + ...+ a_n + k$ objects, where a_1 are alike of one kind, ……….a_n are alike of nth kind and k are distinct is

   $$(a_1 + 1)(a_2 + 1) ....(a_n + 1)\, 2^k - 1.$$

7 DIVISORS OF A GIVEN NATURAL NUMBER

Let $n \in \boldsymbol{N}$ and $n = p_1^{\alpha_1}.\, p_2^{\alpha_2}.\, p_3^{\alpha_3} ......p_k^{\alpha_k}$, where $p_1, p_2, p_3, ... p_k$ are different prime numbers and $\alpha_1, \alpha_2, \alpha_3, ... , \alpha_k$ are natural numbers then:

- the total number of divisors of N including 1 and n is
  $= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) .... (\alpha_k + 1)$
- the total number of divisors of n excluding 1 and n is
  $= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) .... (\alpha_k + 1) - 2$
- the total number of divisors of n excluding exactly one out of 1 or n is

$= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_k + 1) - 1$

- the sum of these divisors is

  $= (p_1^0 + p_1^1 + p_1^2 + \ldots + p_1^{\alpha_1})(p_2^0 + p_2^1 + p_2^2 + \ldots + p_2^{\alpha_2}) \ldots (p_k^0 + p_k^1 + p_k^2 + \ldots + p_k^{\alpha_k})$

  (Use sum of G.*P*. in each bracket)

- the number of ways in which n can be resolved as a product of two factors is

  $\frac{1}{2}(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1)$, if n is not a perfect square

  $\frac{1}{2}[(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1) + 1]$, if n is a perfect square

- the number of ways in which is composite number n can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{k-1} where k is the number of different factors (or different primes) in n.

8 DIVISION OF DISTINCT OBJECTS INTO GROUPS

In the case of grouping we have the following. If $m + n + p$ things are divided into 3 groups one containing m, the second n and the third p things; number of groupings is ${}^{(m+n+p)}C_m \cdot {}^{(n+p)}C_n \cdot {}^{p}C_p$

$= \frac{(m+n+p)!}{m!\,n!\,p!}$ where m, n, p are distinct natural numbers.

If $m = n = p$ (say) then the number of groupings (unmindful of the order of grouping) is

$$\frac{3m!}{(m!)^3\,3!}$$

Thus if 52 cards be divided into four groups of 13 each, the number of groupings is

$$\frac{52!}{(13!)^4 \cdot 4!}$$

On the other hand, when 52 cards are dealt 13 each to four persons, the number of ways in which this can be done is

$$\frac{52!}{(13!)^4}$$

As another example, if we consider the division of 52 cards into four groups, three groups containing each 16 and the fourth cards, the number of ways in which this can be done is

$$\frac{52!}{3!(16!)^3\,4!}$$

Note the 3 ! factor in the denominator. This is for the reason that there are only 3 equal groups.

In general, the number of ways in which mn different things can be divided equally into m distinct groups is $\frac{(mn)!}{(n!)^m}$ when order of groups is important.

Whereas when order of groups is not important, then division into m equal group is done in $\frac{(mn)!}{m!\,(n!)^m}$ ways.

In general, we can write formula for grouping as factorial of total number of elements divided by product of factorial of number of elements in each group and product of factorial of number of groups having same number of elements, if any. Also formula for number of distribution is number of grouping multiplied with factorial of number of persons in which objects are distributed, divided by factorial of number of persons who got nothing, if any.

9 DIVISION OF IDENTICAL OBJECTS INTO GROUPS

The number of ways of division or distribution of n identical things into r different groups is ${}^{n+r-1}C_{r-1}$ or ${}^{n-1}C_{r-1}$ according as empty groups are allowed or not allowed.

10 ARRANGEMENT IN GROUPS

The number of ways of distribution and arrangement of n distinct things into r different groups is $n!\ {}^{n+r-1}C_{r-1}$ or $n!\ {}^{n-1}C_{r-1}$ according as empty groups are allowed or not allowed.

Illustration 33

Question: **In how many ways can three balls of different colours be put in 4 glass cylinders of equal width such that any glass cylinder may have either 0, 1, 2 or 3 balls ?**

Solution: There are four glass cylinders. Consider additionally (4 – 1) = 3 things; and, the number of ways is ${}^6C_3$ (corresponding to ${}^{(n+r-1)}C_{r-1}$) multiplied by 3! = 120.

11 METHODS OF INCLUSION EXCLUSION

If $A_1, A_2, \ldots\ldots, A_m$ are finite sets and $A = A_1 \cup A_2 \cup \ldots \cup A_m$, then

$$n(A) = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{m+1} a_m$$

where $a_1 = n(A_1) + n(A_2) + \ldots + n(A_m)$

$$a_2 = \sum_{1\le i<j\le m} n(A_i \cap A_j)$$

$$a_3 = \sum_{1\le i<j<k\le m} n(A_i \cap A_j \cap A_k)$$

and so on.

Corollary (Sieve-Formula)

If $A_1, A_2, \ldots A_m$ are m subsets of a set A containing N elements, then $n(A_1' \cap A_2' \cap \ldots \cap A_m')$

$$= N - \sum_i n(A_i) + \sum_{1\le i<j\le m} n(A_i \cap A_j) - \sum_{1\le i<j<k\le m} n(A_i \cap A_j \cap A_k)$$

$$+ \ldots + (-1)^m n(A_1 \cap A_2 \cap \ldots \cap A_m)$$

11.1 DEARRANGEMENTS

It is rearrangement of objects such that no one goes to its original place.

If n things are arranged in a row, the number of ways in which they can be rearranged so that none of them occupies its original placed is $n!\left(1-\frac{1}{1!}+\frac{1}{2!}\ldots + (-1)^n \frac{1}{n!}\right)$

It is denoted by $D_n = n!\sum_{r=0}^{n}(-1)^r \frac{1}{r!}$

12 USE OF MULTINOMIALS

- If there are l objects of one kind, m objects of second kind, n objects of third kind and so on; then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + \ldots$) is the coefficient of x^r in the expansion of

$$(1 + x + x^2 + x^3 + \ldots.\, x^l)\,(1 + x + x^2 + \ldots.. + x^m)\,(1 + x + x^2 + \ldots.. + x^n)$$

Further if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + \ldots.$) is the coefficient of x^r in the expansion of

$$(x + x^2 + x^3 + \ldots.\, x^l)\,(x + x^2 + x^3 + \ldots.. + x^m)\,(x + x^2 + x^3 + \ldots.. + x^n)$$

- If there are l objects of one kind, m object of second kind, n object of third kind and so on; then the number of possible arrangements/permutations of r objects out of these objects (i.e., $l + m + n + \ldots..$) is the coefficient of x^r in the expansion of

$$r!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots.+\frac{x^l}{l!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots.+\frac{x^m}{m!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots.+\frac{x^n}{n!}\right)\ldots$$

Note: For use in problems of the above type the following Binomial expansions may be noted

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + \ldots + x^r + \ldots$$

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + \ldots + (r+1)\,x^r + \ldots$$

$$\frac{1}{(1-x)^3} = (1-x)^{-3} = 1 + 3x + 6x^2 + \ldots + \frac{(r+1)(r+2)}{1\cdot 2}x^r + \ldots..$$

$$\frac{1}{(1-x)^4} = (1-x)^{-4} = 1 + 4x + 10x^2 + \ldots + \frac{(r+1)(r+2)(r+3)}{1\cdot 2\cdot 3}x^r + \ldots...$$

and more generally

$$\frac{1}{(1-x)^p} = (1-x)^{-p} = 1 + px + \ldots + \frac{(r+1)(r+2)\ldots(r+\overline{p-1})}{1\cdot 2\cdot 3\ \ldots\ (p-1)}\,x^r + \ldots$$

and the coefficient of x^r in this general case is easily seen to be ${}^{(r+p-1)}C_{p-1}$

13 USE OF MULTINOMIALS IN SOLVING LINEAR EQUATION

Multinomials can be used to solve linear equations. The method is illustrated with the help of following illustrations.

14 EXPONENT OF PRIME p IN n!

Since, exponent of prime p in $n!$ is denoted by $E_p\,(n!)$, where p is a prime number and n is a natural number. Then the last integer amongst 1, 2, 3, ……, $(n-1)$, n which is divisible by p is $[n/p]\,p$, where $[n/p]$ denotes the greatest integer. $(\because\ [x] \le x)$

$$E_p(n!) = E_p(1, 2, 3 \ldots..(n-1)\,.\,n)$$
$$= E_p(p\,.\,2p\,.\,3p\,.\,\ldots..\,(n-1)\,p\,.\,[n/p]\,p)$$

Because the remaining natural numbers from 1 to n are not divisible by p.

Thus, $\left[\frac{n}{p}\right] + E_p\left(1.2.3\ldots\left[\frac{n}{p}\right]\right)$

Now the last integer among 1, 2, 3, ….. , $\left[\frac{n}{p}\right]$ which is divisible by p is

$$\left[\frac{n/p}{p}\right] = \left[\frac{n}{p^2}\right] = \left[\frac{n}{p}\right] + E_p\left(p.2p.3\ldots..\left[\frac{n}{p^2}\right]p\right)$$

Because the remaining natural numbers from 1 to $\left[\frac{n}{p}\right]$ are not divisible by p.

Thus, $\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+E_p\left(1.2.3...\left[\frac{n}{p^2}\right]\right)$

Similarly, we get $E_p(n!)=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+......+\left[\frac{n}{p^s}\right]$

Where s is the largest natural number such that $p^s \le n < p^{s+1}$

Note : $E_{rp}(n!) = E_r(n!)$ (If $r > p$)

Where r and p are prime numbers.

15 PROBLEMS ON FORMATION OF NUMBERS

Formation of numbers using ten digits with or without repetition is also one of the counting techniques or combinatorics some illustrations as follows:

MIND MAP

PERMUTATIONS & COMBINATIONS

FUNDAMENTAL PRINCIPLE OF COUNTING

- **Multiplication principle:**
  If a job can be completed in m ways and another in n ways then both can be completed in mn ways.
- **Addition principle:**
  If a job can be completed in m ways and another in n ways then either of them can be completed in $(m + n)$ ways.

COMBINATIONS

- It is the numbers of ways of selection of r objects from n distinct objects
  $${}^nC_r = \frac{n!}{(n-r)!\,r!} \qquad (0 \le r \le n)$$
- ${}^nC_0 = {}^nC_n = 1$
- ${}^nC_x = {}^nC_y = \Rightarrow x = y$ or $x + y = n$
- The greatest value of ${}^nC_r$ is ${}^nC_{n/2}$ if n is even, ${}^nC_{(n-1)/2}$ or ${}^nC_{(n+1)/2}$ if n is odd.
- The number of ways in which r objects can be selected from n distinct objects
  (i) if a particular object is always included is ${}^{n-1}C_{r-1}$ (ii) if a particular object is always excluded is ${}^{n-1}C_r$ (iii) if m particular objects are always included is ${}^{n-m}C_{r-m}$ (iv) if m particular objects are always excluded is ${}^{n-m}C_r$.

SELECTION FROM IDENTICAL/DISTINCT OBJECTS

- The number of ways of selection of atleast one from n distinct objects is
  $${}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n - 1.$$
- The number of ways of selection of atleast one from $a_1 + a_2 + \ldots + a_n$ objects, where a_i $(i = 1, 2, \ldots n)$ are identical of some kind is $(a_1 + 1)(a_2 + 1) \ldots (a_n + 1) - 1$.
- $a_1 + a_2 + \ldots a_n + k$, where a_i's are alike of some kind and k are distinct is
  $$(a_1 + 1)(a_2 + 1) \ldots (a_n + 1)\, 2^k - 1.$$

DIVISORS OF GIVEN NATURAL NUMBERS

- Let $N = p_1^{\alpha_1}.\, p_2^{\alpha_2}.\, p_3^{\alpha_3} \ldots\ldots p_k^{\alpha_k}$, where $p_1, p_2, p_3, \ldots p_k$ are different prime numbers and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are natural numbers then:
  the total number of divisors of N including 1 and N is $= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_k + 1)$
- the total number of divisors of N excluding 1 and N is $= (\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1) - 2$
- the sum of these divisors is =
  $$(p_1^0 + p_1^1 + \ldots + p_1^{\alpha_1}) \ldots (p_k^0 + p_k^1 + \ldots + p_k^{\alpha_k})$$
  (Use sum of G.P. in each bracket)

EXPONENT OF PRIME p IN $n!$

- $E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots \left[\frac{n}{p^s}\right]$
  where s is the largest natural number such that $p^s \le n \le p^{s+1}$.
- $E_{rp}(n!) = E_r(n!)$ (if $r > p$) where r and p both are prime numbers.

PERMUTATIONS

- The number of linear arrangements of r objects out of n distinct objects, is denoted as ${}^nP_r$, where ${}^nP_r = \frac{n!}{(n-r)!}$.
- The number of permutations of n different things taken r at a time when each thing may be repeated any number of times is n^r.
- The number of arrangements of n things, when p out of them are of one kind, q are of other kind and r of another kind and $p + q + r \le n$ is $\frac{n!}{p!\,q!\,r!}$.

CIRCULAR PERMUTATIONS

- n distinct things can be arranged in a circular in $(n - 1)!$ ways.
- n distinct beads can be arranged in a garland in ½ [(n – 1)]!

DIVISION OF DISTINCT OBJECTS INTO GROUPS

- mn different things can be distributed in m equal groups of n objects each, in
  (i) $\frac{mn!}{(n!)^m}$ ways, when order of groups is important.
  (ii) $\frac{(mn)!}{m!\,(n!)^n}$ ways, when order of groups is not important.

DIVISION OF IDENTICAL OBJECTS INTO GROUPS

- n identical objects can be divided into r different groups in ${}^{n+r-1}C_{r-1}$ or ${}^{n-1}C_{r-1}$ according as blank groups are not considered.

ARRANGEMENT IN GROUPS

- The number of ways of distribution and arrangement of n distinct things into r different groups is ${}^{n+r-1}C_{r-1}$ or $n!\ {}^{n-1}C_{r-1}$ as blank groups are not considered.

DEARRANGEMENT

- n things arrangement in a row can be dearranged in $n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \ldots (-1)^n \frac{1}{n!}\right)$ ways such that none of them occupies its original place.

USE OF MULTINOMIALS

- If there are l objects of one kind, m objects of second kind, n objects of third kind and so on; then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + \ldots$) is the coefficient of x^r in $(1 + x + x^2 + x^3 + \ldots x^l)(1 + x + x^2 + \ldots + x^m)(1 + x + x^2 + \ldots + x^n)$. Further if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects is the coefficient of x^r in $(x + x^2 + x^3 + \ldots x^l)(x + x^2 + x^3 + \ldots + x^m)(x + x^2 + x^3 + \ldots + x^n)$ and the number of possible permutations of r objects out of these objects is the coefficient of x^r in
  $$r!\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^l}{l!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^m}{m!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}\right)\ldots$$
  While applying it we use the coefficient of x^r in $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$ or ${}^{n+r-1}C_{n-1}$.