

## Mathematical Induction

### 1 INTRODUCTION

The word induction means the method of reasoning about a general statement from the conclusion of particular cases. Inductions starts with observations. It may be true but then it must be so proved by the process of reasoning. Else it may be false but then it must be shown by finding a counter example where the conjecture fails.

In mathematics there are some results or statements that are formulated in terms of  $n$ , where  $n \in N$ . To prove such statements we use a well suited method, based on the specific technique, which is known as **principle of mathematical induction**.

### 2 PROPOSITION

A statement which is either true or false is called a proposition or statement.

$P(n)$  denotes a proposition whose truth value depends on natural variable ' $n$ '.

For example  $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$  is a proposition whose truth value depends on natural number  $n$ .

We write  $P(n): 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ ,

where  $P(4)$  means  $1^2 + 2^2 + 3^2 + 4^2 = \frac{4(4+1)(8+1)}{6}$

To prove the truth of proposition  $P(n)$  depending on natural variable  $n$ , we use mathematical induction.

### 3 FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

The statement  $P(n)$  is true for all  $n \in N$  if

- (i)  $P(1)$  is true.
- (ii)  $P(m)$  is true  $\Rightarrow P(m+1)$  is true.

The above statement can be generalized as  $P(n)$  is true for all  $n \in N$  and  $n \geq k$  if

- (i)  $P(k)$  is true.
- (ii)  $P(m)$  is true ( $m > k$ )  $\Rightarrow P(m+1)$  is true.

### 4 APPLICATION OF FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

#### WORKING RULE

To prove any statement  $P(n)$  to be true for all  $n \geq k$  with the help of first principle of mathematical induction we follow the following procedure:

- Step (i) Check if the statement is true or false for  $n = k$ .
- Step (ii) Assume the statement is true for  $n = m$ .
- Step (iii) Prove the statement is true for  $n = m + 1$ .



### Illustration 1

**Question:** Prove by the principle of mathematical induction that for all  $n \in \mathbb{N}$ :

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

**Solution:** Let  $P(n)$  be the statement given by

$$P(n) : \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Step I: We have,  $P(1) : \frac{1}{1.2} = \frac{1}{1+1}$

Since,  $\frac{1}{1.2} = \frac{1}{1+1} = \frac{1}{2}$

So,  $P(1)$  is true.

Step II: Let  $P(m)$  be true, then  $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{m(m+1)} = \frac{m}{m+1}$  (i)

We shall now show that  $P(m+1)$  is true. If  $P(m)$  is true.

For this we have to show that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+1+1)} = \frac{m+1}{(m+1)+1}$$

Now, 
$$\begin{aligned} & \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+1+1)} \\ &= \left[ \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{m(m+1)} \right] + \frac{1}{(m+1)(m+1+1)} \\ &= \frac{m}{m+1} + \frac{1}{(m+1)((m+1)+1)} \quad \text{[using (i)]} \\ &= \frac{1}{(m+1)} \left\{ \frac{m}{1} + \frac{1}{m+2} \right\} = \frac{1}{(m+1)} \frac{(m^2 + 2m + 1)}{(m+2)} = \frac{(m+1)^2}{(m+1)(m+2)} = \frac{m+1}{m+2} \end{aligned}$$

$\therefore P(m+1)$  is true.

Thus,  $P(m)$  is true  $\Rightarrow P(m+1)$  is true.

Hence, by the principle of mathematical induction, the given statement is true for all  $n \in \mathbb{N}$ .

### Illustration 2

**Question:** For every positive integer  $n$ , prove that  $7^n - 3^n$  is divisible by 4.

**Solution:** We have  $P(n) : 7^n - 3^n$  is divisible by 4

We note that  $P(1) : 7^1 - 3^1 = 4$ , which is divisible by 4. Thus  $P(n)$  is true for  $n = 1$

Let  $P(k)$  be true for some natural number  $k$ .

i.e.,  $P(k) : 7^k - 3^k$  is divisible by 4.

We get  $7^k - 3^k = 4d$ , where  $d \in \mathbb{N}$  ... (i)

Now, we wish to prove that  $P(k+1)$  is true whenever  $P(k)$  is true.

i.e., we have show  $7^{k+1} - 3^{k+1} = 4m$

$$\begin{aligned} \text{Now, } 7^{(k+1)} - 3^{(k+1)} &= 7^{(k+1)} - 7 \cdot 3^k + 7 \cdot 3^k - 3^{(k+1)} \\ &= 7(7^k - 3^k) + (7 - 3)3^k = 7(4d) + (7 - 3)3^k \end{aligned}$$

$$= 7(4d) + 4 \cdot 3^k \quad [\text{using (i)}]$$

$$= 4(7d + 3^k) = 4m$$

From the last line, we see that  $7^{(k+1)} - 3^{(k+1)}$  is divisible by 4.

Thus,  $P(k+1)$  is true when  $P(k)$  is true.

Therefore, by principle of mathematical induction the statement is true for every positive integer  $n$ .

### Illustration 3

**Question:** Prove that  $2 \cdot 7^n + 3 \cdot 5^n - 5$  is divisible by 24 for all  $n \in \mathbf{N}$ .

**Solution:** Let the statement  $P(n)$  be defined as

$P(n)$ :  $2 \cdot 7^n + 3 \cdot 5^n - 5$  is divisible by 24.

Now,  $P(n)$  is true for  $n = 1$ , since  $2 \cdot 7 + 3 \cdot 5 - 5 = 24$ , which is divisible by 24.

Assume that  $P(k)$  is true. i.e.  $2 \cdot 7^k + 3 \cdot 5^k - 5 = 24q$ , when  $q \in \mathbf{N}$  ... (i)

Now, we wish to prove that  $P(k+1)$  is true whenever  $P(k)$  is true.

$$\begin{aligned} \text{We have, } 2 \cdot 7^{k+1} + 3 \cdot 5^{k+1} - 5 &= 2 \cdot 7^k \cdot 7^1 + 3 \cdot 5^k \cdot 5^1 - 5 \\ &= 7[2 \cdot 7^k + 3 \cdot 5^k - 5 - 3 \cdot 5^k + 5] + 3 \cdot 5^k \cdot 5 - 5 \\ &= 7[24q - 3 \cdot 5^k + 5] + 15 \cdot 5^k - 5 \quad [\text{using (i)}] \\ &= 7 \times 24q - 21 \cdot 5^k + 35 + 15 \cdot 5^k - 5 \\ &= 7 \times 24q - 6 \cdot 5^k + 30 \\ &= 7 \times 24q - 30(5^{k-1} - 1) = 7 \times 24q - 30(4p) \\ &[\text{since } (5^{k-1} - 1) \text{ is a multiple of 4 as } x^n - y^n \text{ is divisible by } x - y] \\ &= 7 \times 24q - 120p = 24(7q - 5p) \\ &= 24 \times r, \quad r = 7q - 5p, \text{ is some natural number} \quad \dots \text{(ii)} \end{aligned}$$

The expression on the R.H.S. of (ii) is divisible by 24.

Thus  $P(k+1)$  is true whenever  $P(k)$  is true.

Hence, by principle of mathematical induction,  $P(n)$  is true for all  $n \in \mathbf{N}$

### Illustration 4

**Question:** Prove the rule of exponents  $(ab)^n = a^n b^n$  by using principle of mathematical induction for every natural number.

**Solution:** Let  $P(n)$  be the given statement i.e.  $P(n)$ :  $(ab)^n = a^n b^n$

We note that  $P(n)$  is true for  $n = 1$  since  $(ab)^1 = a^1 b^1$

Let  $P(k)$  be true, i.e.,  $(ab)^k = a^k b^k$  ... (i)

We shall now prove that  $P(k+1)$  is true whenever  $P(k)$  is true.

$$\begin{aligned} \text{Now, we have } (ab)^{k+1} &= (ab)^k (ab) \\ &= (a^k b^k) (ab) \quad [\text{by (i)}] \\ &= (a^k \cdot a^1) (b^k \cdot b^1) = a^{k+1} \cdot b^{k+1} \end{aligned}$$

Therefore,  $P(k+1)$  is also true whenever  $P(k)$  is true.

Hence, by principle of mathematical induction,  $P(n)$  is true for all  $n \in \mathbf{N}$ .

### Illustration 5

**Question:** Using mathematical induction, show that

$$\cos \theta \cos 2\theta \cos 4\theta \dots \cos(2^{n-1}\theta) = \frac{\sin 2^n \theta}{2^n \sin \theta}, \quad \forall n \in \mathbf{N}.$$



**Solution:** Let  $P(n): \cos \theta \cos 2\theta \cos 4\theta \dots \cos(2^{n-1}\theta) = \frac{\sin 2^n \theta}{2^n \sin \theta}$

Step I: For  $n = 1$

L.H.S. of (i) =  $\cos \theta$  and R.H.S. of (i) =  $\frac{\sin 2\theta}{2 \sin \theta} = \cos \theta$

Therefore,  $P(1)$  is true.

Step II: Assume it is true for  $n = k$ , then

$$p(k): \cos \theta \cos 2\theta \cos 4\theta \dots \cos(2^{k-1}\theta) = \frac{\sin 2^k \theta}{2^k \sin \theta} \quad \dots(i)$$

Step III: For  $n = k+1$

$$P(k+1): \cos \theta \cos 2\theta \cos 4\theta \dots \cos(2^{k-1}\theta) \cos(2^k \theta) = \frac{\sin 2^{k+1} \theta}{2^{k+1} \sin \theta}$$

$$\begin{aligned} \text{L.H.S.} &= \cos \theta \cos 2\theta \cos 4\theta \dots \cos(2^{k-1}\theta) \cos(2^k \theta) \\ &= \frac{\sin(2^k \theta)}{2^k \sin \theta} \cdot \cos(2^k \theta) = \frac{2 \sin(2^k \theta) \cos(2^k \theta)}{2^{k+1} \sin \theta} \quad [\text{using (i)}] \\ &= \frac{\sin(2 \cdot 2^k \theta)}{2^{k+1} \sin \theta} = \frac{\sin(2^{k+1} \theta)}{2^{k+1} \sin \theta} = \text{R.H.S.} \end{aligned}$$

This shows that the  $P(k+1)$  is true if  $P(k)$  is true.

Hence by the principle of mathematical induction, the result is true for all  $n \in N$ .

### Illustration 6

**Question:** Prove by induction that the sum  $S_n = n^3 + 3n^2 + 5n + 3$  is divisible by 3 for all  $n \in N$ .

**Solution:** Let  $P(n)$  be the statement given by

$$P(n): S_n = n^3 + 3n^2 + 5n + 3 \text{ is divisible by 3}$$

Step I: We have,  $P(1): S_1 = 1^3 + 3(1)^2 + 5(1) + 3$  is divisible by 3

Since  $1^3 + 3(1)^2 + 5(1) + 3 = 12$ , which is divisible by 3

$\therefore P(1)$  is true.

Step II: Let  $P(m)$  be true. Then

$$S_m = m^3 + 3m^2 + 5m + 3 \text{ is divisible by 3}$$

$$\Rightarrow S_m = m^3 + 3m^2 + 5m + 3 = 3\lambda, \text{ for some } \lambda \in N \quad \dots(i)$$

We now wish to show that  $P(m+1)$  is true. For this we have to show that

$$(m+1)^3 + 3(m+1)^2 + 5(m+1) + 3 \text{ is divisible by 3}$$

$$\begin{aligned} \text{Now, } (m+1)^3 + 3(m+1)^2 + 5(m+1) + 3 &= (m^3 + 3m^2 + 5m + 3) + 3m^2 + 9m + 9 \\ &= 3\lambda + 3(m^2 + 3m + 3) = 3[\lambda + m^2 + 3m + 3] \quad [\text{using (i)}] \end{aligned}$$

$$= 3\mu, \text{ where } \mu = \lambda + m^2 + 3m + 3 \in N$$

$\therefore P(m+1)$  is true.

Thus,  $P(m)$  is true  $\Rightarrow P(m+1)$  is true

Hence, by the principle of mathematical induction the statement is true for all  $n \in N$ .

### Illustration 7

**Question:** Show by using principle of mathematical induction that

$$1.3 + 2.3^2 + 3.3^3 + \dots + n.3^n = \frac{(2n-1)3^{n+1} + 3}{4}$$

**Solution:** Let  $P(n): 1.3 + 2.3^2 + 3.3^3 + \dots + n.3^n = \frac{(2n-1)3^{n+1} + 3}{4}$

Step I: When  $n = 1$ , L.H.S. =  $1.3 = 3$

$$\text{and R.H.S.} = \frac{(2n-1)3^{n+1} + 3}{4} = \frac{(2 \cdot 1 - 1)3^2 + 3}{4} = \frac{12}{4} = 3$$

Hence  $P(1)$  is true.

Let  $P(m)$  be true

$$\Rightarrow 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \dots + m \cdot 3^m = \frac{(2m-1)3^{m+1} + 3}{4} \quad \dots(i)$$

To prove  $P(m+1)$  is true i.e.,

$$1 \cdot 3 + 2 \cdot 3^2 + \dots + m \cdot 3^m + (m+1) \cdot 3^{m+1} = \frac{(2m+1)3^{m+2} + 3}{4}$$

Adding  $(m+1) \cdot 3^{m+1}$  to both sides of (i), we get

$$\begin{aligned} 1 \cdot 3 + 2 \cdot 3^2 + \dots + m \cdot 3^m + (m+1) \cdot 3^{m+1} \\ &= \frac{(2m-1)3^{m+1} + 3}{4} + (m+1) \cdot 3^{m+1} \\ &= \frac{\{2m-1+4(m+1)\} \cdot 3^{m+1} + 3}{4} = \frac{(2m+1)3^{m+2} + 3}{4} \end{aligned}$$

Hence  $P(m+1)$  is true whenever  $P(m)$  is true.

It follows that  $P(n)$  is true for all natural numbers  $n$ .

### Illustration 8

**Question:** Using the principle of mathematical induction, show that  $11^{n+2} + 12^{2n+1}$ , where  $n$  is a natural number, is divisible by 133.

**Solution:** Let  $f(n) = 11^{n+2} + 12^{2n+1}$

Let  $P(n): f(n)$  i.e.  $11^{n+2} + 12^{2n+1}$  is divisible by 133

$$\begin{aligned} \text{Now, } f(1) &= 11^3 + 12^3 = (11+12)(121-11 \times 12+144) \\ &= 23 \times 133, \text{ where is divisible by 133} \end{aligned}$$

$\therefore P(1)$  is true. ...(A)

Let  $P(m)$  be true  $\Rightarrow f(m) = 11^{m+2} + 12^{2m+1}$  is divisible by 133

$$\Rightarrow f(m) = 11^{m+2} + 12^{2m+1} = 133k, \text{ where } k \text{ is an integer} \quad \dots(i)$$

$$\begin{aligned} \text{Now, } f(m+1) &= 11^{m+3} + 12^{2m+3} \\ &= 11 \cdot 11^{m+2} + 12^{2m+1} \cdot 12^2 = 11(11)^{m+2} + 12^{2m+1} \cdot 144 \end{aligned}$$

Now we divide  $f(m+1)$  by  $f(m)$

$$\begin{array}{r} 11^{m+2} + 12^{2m+1} \bigg| 11 \cdot (11)^{m+2} + 144 \cdot 12^{2m+1} \\ \underline{11 \cdot (11)^{m+2} + 11 \cdot 12^{2m+1}} \\ \hline 133 \cdot 12^{2m+1} \end{array}$$

$$\begin{aligned} \therefore f(m+1) &= 11 \cdot f(m) + 133 \cdot 12^{2m+1} = 11 \times 133k + 133 \cdot 12^{2m+1} \quad [\text{from (i)}] \\ &= 133(11k + 12^{2m+1}), \text{ which is divisible by 133} \end{aligned}$$

$\therefore P(m+1)$  is true whenever  $P(m)$  is true ...(B)

From (A) and (B) it follows that  $P(n)$  is true for every natural number  $n$  i.e.  $11^{n+2} + 12^{2n+1}$  is divisible by 133 for every natural number  $n$ .

### Illustration 9

**Question:** Prove that  $1^2 + 2^2 + \dots + n^2 > \frac{n^3}{3}$ ,  $n \in \mathbb{N}$ .

**Solution:** Let  $P(n)$  be the given statement.

i.e.,  $P(n): 1^2 + 2^2 + \dots + n^2 > \frac{n^3}{3}, n \in N$

when note that  $P(n)$  is true for  $n = 1$  since  $1^2 > \frac{1^3}{3}$

Assume that  $P(k)$  is true

i.e.,  $P(k): 1^2 + 2^2 + \dots + k^2 > \frac{k^3}{3} \dots(i)$

We shall now prove that  $P(k+1)$  is true whenever  $P(k)$  is true.

$$\begin{aligned} \text{We have, } 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 \\ &= (1^2 + 2^2 + \dots + k^2) + (k+1)^2 > \frac{k^3}{3} + (k+1)^2 \quad [\text{by (i)}] \\ &= \frac{1}{3}[k^3 + 3k^2 + 6k + 3] \\ &= \frac{1}{3}[(k+1)^3 + 3k + 2] > \frac{1}{3}(k+1)^3 \end{aligned}$$

Therefore,  $P(k+1)$  is also true whenever  $P(k)$  is true.

Hence, by mathematical induction  $P(n)$  is true for all  $n \in N$

### Illustration 10

**Question:** Prove that  $2^n > n$  for all positive integers  $n$ .

**Solution:** Let  $P(n): 2^n > n$

When  $n = 1, 2^1 > 1$

Hence  $P(1)$  is true.

Assume that  $P(k)$  is true for any positive integers  $k$  i.e.,

$2^k > k \dots(i)$

We shall now prove that  $P(k+1)$  is true whenever  $P(k)$  is true.

Multiplying both sides of (i) by 2, we get

$$2 \cdot 2^k > 2k$$

i.e.,  $2^{k+1} > 2k = k + k > k + 1$

$\therefore P(k+1)$  is true when  $P(k)$  is true.

Hence, by principle of mathematical induction,  $P(n)$  is true for every positive integer  $n$ .