RELATIONS AND FUNCTIONS

1 RELATIONS

1.1 INTRODUCTION

In our day to day life, we often talk about relation between two persons, between two straight lines (e.g. perpendicular lines, parallel lines) etc.

Let A be the set of all male students in Delhi whose fathers live in Delhi. Let B be the set of all the people living in Delhi. Let a be a male student living in Delhi i.e. $a \in A$. Let b be the father of a. Then $b \in B$. And a is related to b under son-father relation. If we denote the son-father relation by symbol R then a is related to b under relation R. We can also express this by writing aRb. Here R denotes the relation 'is son of'.

We can also express this statement by saying that the pair of a and b is in relation R i.e., the ordered pair $(a, b) \in R$. This pair (a, b) is ordered in the sense that a and b can't be interchanged because first co-ordinate a represents son, and the second coordinate b represents father of a. Similarly if $a_1 \in A$ and b_1 is father of a_1, then $(a_1, b_1) \in R$. So we can think of the relation R as a set of ordered pairs whose first coordinate is in A and the second coordinate is in B. Thus $R \subseteq A \times B$. Since the relation 'is son of' i.e., R is a relation relating elements of A to be elements of B, we will say that R is a relation from set A to set B.

1.2 DEFINITION

A relation R, from a non-empty set A to another non-empty set B, is a subset of $A \times B$
Equivalently, any subset of $A \times B$ is relation from A to B.
Thus, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$

$$\Leftrightarrow R \subseteq \{(a, b) : a \in A, \ b \in B\}$$

Example: Let $A = \{1, 2\}, B = \{a, b, c\}$
Let $R = \{(1, a), (1, c)\}$
Here R is a subset of $A \times B$ and hence it is a relation from A to B.

2 DOMAIN AND RANGE OF A RELATION

2.1 DOMAIN OF A RELATION

Let R be a relation from A to B. The domain of relation R is the set of all those elements $a \in A$ such that $(a, b) \in R$ for some $b \in R$. Domain of R is precisely written as domain R.

Thus domain of (R) = $\{a \in A : (a, b) \in R \text{ for some } b \in B\}$

Thus domain of R = set of first components of all the ordered pair which belong to R.

2.2 RANGE OF A RELATION

Let R be a relation from A to B. The range of R is the set of all those elements $b \in R$ such that $(a, b) \in R$ for some $a \in A$.

Thus range of $R = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$.

Range of R = set of second components of all the ordered pairs which belong to R.
Set B is called as codomain of relation R.

Example1: Let $A = \{2, 3, 5\}$ and $B = \{4, 7, 10, 8\}$
Let $aRb \Leftrightarrow a$ divides b
Then $R = (2, 5)$ and range of $R = \{4, 10, 8\}$
Codomain of $R = B = \{4, 7, 10, 8\}$

Example2: Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8\}$

Let R be a relation defined from A to B by $xRy \Leftrightarrow y$ is double of $x, \forall\ x \in A$

Then $1R2,\ 2R4,\ 3R6$

$\therefore\ R = \{(1, 2), (2, 4), (3, 6)\}$

3 REPRESENTATION OF A RELATION

A relation from a set A to set B can be represented in any one of the following four forms.

3.1 ROSTER FORM

In this form a relation R is represented by the set of all ordered pairs belonging to R.

Example: Let A = {–1, 1, 2} and B = {1, 4, 9, 10}

Let aRb means $a^2 = b$

Then R (in roaster form) = {(–1, 1), (1, 1), (2, 4)}

3.2 SET-BUILDER FORM

In this form, the relation R is represented as $\{(a, b) : a \in A, b \in B, a.....b\}$, the blank is to be replaced by the rule which associates a to b.

Example: Let A = {1, 3, 5, 7}, B = {2, 4, 6, 8}

Let R = {(1, 2), (3, 4), (5, 6), (7, 8)}, then R in the builder form can be written as

$R = \{(a, b) : a \in A, b \in B; a - b = -1\}$

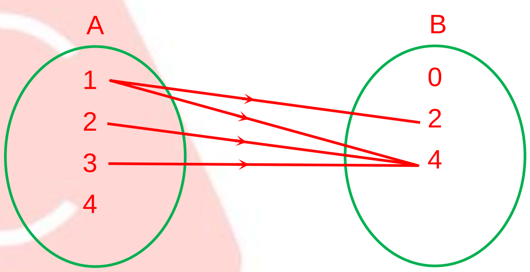
3.3 BY ARROW DIAGRAM

In this form, the relation R is represented by drawing arrows from first component to the second component of all ordered pairs belonging to R.

Example: Let A = {1, 2, 3, 4}, B = {0, 2, 4} and R be relation 'is less than' from A to B, then

$R = \{(1, 2), (1, 4), (2, 4), (3, 4)\}$

This relation R from A to B can be represented by the arrow diagram as shown in the figure.



4 TOTAL NUMBER OF RELATIONS

Let A and B be two non empty finite sets having p and q elements respectively.

Then $n(A \times B) = n(A).n(B) = pq$

Therefore, total number of subsets of $A \times B = 2^{pq}$

Since each subset of $A \times B$ is a relation from A and B, therefore total number of relations form A to B is 2^{pq}

Note: Empty relation ϕ and universal relation $A \times B$ are called trivial relations and any other relation is called a non trivial relation.

Example: Let A = {1, 2}, B = {3, 4, 5}

Then $n(A \times B) = n(A).n(B) = 2 \times 3 = 6$

$\therefore$ Number of relations from A to B = 2^6 = 64

Important formulae/points

- *If R is relation from A to B and (a, b) $\notin$ R, then we also write a $\not R$ b (read as a is not related to b)*
- *In an identity relation on A every element of A should be related to itself only.*
- *aRb shows that a is the element of domain set and b is the element of range set.*

5 FUNCTIONS

The concept of functions is very important because of its close relation with various phenomena of reality. Thus when we square a given real number in fact we perform an operation on the number x to get number x^2. Hence a function may be viewed as a rule which produces new elements from some given elements. Function is also called mapping or map.

- **Independent Variable**

  The symbol which can take an arbitrary value from a given set is called an independent variable.

- **Dependent Variable**

  The symbol whose value depends on independent variables is called a dependent variable.

6 DEFINITION OF A FUNCTION

- **Definition 1**

  A function *f* is a relation from a non-empty set *A* to a non-empty set *B* such that domain of *f* is *A* and no two distinct ordered pairs in *f* have the same first element.

- **Definition 2**

  Let *A* and *B* be two non-empty sets, then a rule of which associates each element of *A* with a unique element of *B* is called a mapping or a function from *A* to *B* we write $f : A \to B$ (read as *f* is a function from *A* to *B*).

If *f* associates $x \in A$ to $y \in B$, then we say that *y* is the image of the element *x* under the function *f* or the *f* image it by $f(x)$ and we write $y = f(x)$. The element *x* is called the pre-image or inverse-image of *y*.

Thus for a function from *A* to *B*:

(i) *A* and *B* should be non-empty.

(ii) Each element of *A* should have image in *B.*

(iii) No element of *A* should have more than one images in *B*.

7 DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

The set *A* is called as the domain of the map *f* and the set *B* is called as the co-domain. The set of the images of all the elements of *A* under the map *f* is called the range of *f* and is denoted by $f(A)$.

Thus range of *f* i.e. $f(A) = \{f(x) : x \in A\}$.

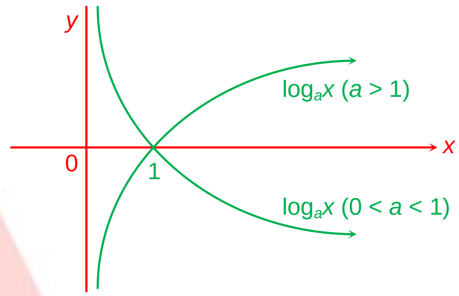
Clearly $f(A) \subseteq B$

Thus,

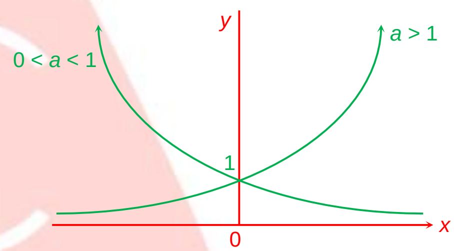
- It is necessary that every *f* image is in *B*, but there may be some elements in *B*, which are not *f* image of any element of *A* i.e., whose pre-image under *f* is not in *A*.
- Two or more elements of *A* may have same image in *B*.
- $f : x \to y$ means that under the function *f* from *A* to *B*, an element *x* of *A* has image *y* in *B*.
- If domain and range of a function are not to be written, sometimes we denote the function *f* by writing $y = f(x)$ and read it as *y* is a function of *x*.
- A function which has *R* or one of its subsets as its range is called “real valued function”. Further, if its domain is also *R* or a subset of *R,* it is called a real function, where *R* is the set of real numbers.

8 IMPORTANT FUNCTIONS AND THEIR GRAPHS

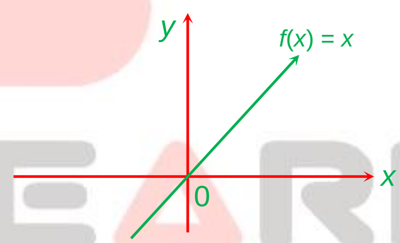
- **Algebraic functions:** Functions consisting of finite number of terms involving powers and roots of the independent variable with the operations +, –, ×, ÷ are called algebraic functions.
  Examples: $f(x)=\sqrt{x-1}$, $f(x)=\sqrt{x}+x^3$
- **Polynomial functions:** $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n$, where $a_0, a_1, a_2, \ldots\ldots, a_n \in \boldsymbol{R}$ is said to be *a* polynomial function of degree *n*.
- **Logarithmic function:** If $a > 0$, $a \neq 1$, then the function $y = \log_a x$, $x \in R^+$ (set of positive real numbers) is called *a* logarithmic function, if $a = e$, the logarithmic function is denoted by $\ln x$.
  Logarithmic function is the inverse of the exponential function.
  For $\log_a x$ to be real, *x* must be greater than zero.
  $y = \log_a x$, $a > 0$ and $\neq 1$
  Domain : $(0, \infty)$; Range : $(-\infty, \infty)$;



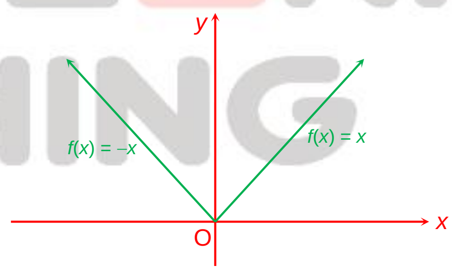
- **Exponential function:** If $a > 0$, $a \neq 1$, then the function defined by $y = a^x$, $x \in R$ is called an exponential function with base *a*.
  $y = f(x) = a^x$, $a > 0$, $a \neq 1$
  Domain : $\boldsymbol{R}$; Range : $(0, \infty)$;



- **Identity function:** An identity function in *x* is defined as $f : \boldsymbol{R} \to \boldsymbol{R}$, $f(x) = x$.



- **Absolute value function:** An absolute value function in *x* is defined as $f : \boldsymbol{R} \to \boldsymbol{R}$, $f(x) = |x|$.
  $$y = f(x) = |x| = \begin{cases} -x, & x \le 0 \\ x, & x > 0 \end{cases}$$
  Domain : $\boldsymbol{R}$; Range : $[0, \infty)$;



Note that $x = 0$ can be included either with positive values of *x* or with negative values of *x*. As we know, all real numbers can be plotted on the real number line, $|x|$ in fact represents the distance of number '*x*' from the origin, measured along the number-line. Thus $|x| \ge 0$. Secondly, any point '*x*' lying on the real number line will have it's coordinates as $(x, 0)$. Thus it's distance from the origin is $\sqrt{x^2}$. Hence $|x|=\sqrt{x^2}$. Thus we can define $|x|$ as $|x|=\sqrt{x^2}$ e.g. if $x = -2.5$, then $|x| = 2.5$, if $x = 3.8$ then $|x| = 3.8$.
There is another way to define $|x|$ as $|x| = \max\{x, -x\}$.

Basic properties of | *x* |

- $|| x || = | x |$
- Geometrical meaning of $|x - y|$ is the distance between x and y.
- $| x | > a \Rightarrow x > a$ or $x < -a$ if $a \in \boldsymbol{R}^+$ and $x \in \boldsymbol{R}$ if $a \in \boldsymbol{R}^-$.
- $| x | < a \Rightarrow -a < x < a$ if $a \in \boldsymbol{R}^+$ and $x \in \phi$ if $a \in \boldsymbol{R}^- \cup \{0\}$
- $| xy | = | x || y |$
- $\left|\dfrac{x}{y}\right| = \dfrac{| x |}{| y |}, y \neq 0$
- $| x + y | \leq | x | + | y |$
  It is a very useful and interesting property. Here the equality sign holds if x and y either both are non-negative or non-positive (i.e. $x.\ y \geq 0$). $(| x | + | y |)$ represents the sum of distances of numbers x and y from the origin and $|x + y|$ represents the distance of number $x + y$ from the origin (or distance between 'x' and '$-y$' measured along the number line).
- $|x - y| \geq | x | \sim | y |$
  Here again the equality sign holds if x and y either both are non-negative or non-positive (i.e. $x.\ y \geq 0$). $(| x | - | y |)$ represents the difference of distances of numbers x and y from the origin and $|x - y|$ represents the distance between 'x' and 'y' measured along the number line.
  The last two properties can be put in one compact form i.e., $|x| \sim |y| \leq |x \pm y| \leq |x| + |y|$.

➢ **Greatest integer function (step function):** The function $f(x) = [x]$ is called the greatest integer function and is defined as follows:

$[x]$ is the greatest integer less than or equal to x .

Then $[x] = x$ if x is an integer

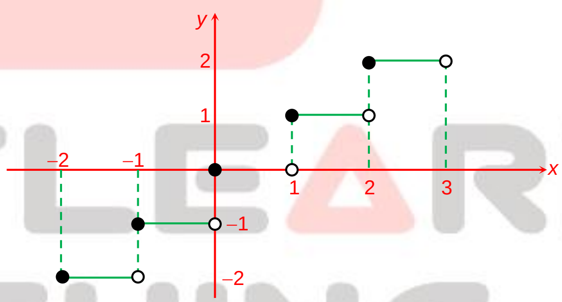
= integer just less than x if x is not an integer.

Examples: $[3] = 3$, $[2.7] = 2$, $[-7.8] = -8$, $[0.8] = 0$

In other words if we list all the integers less then or equal to x, then the integer greatest among them is called greatest integer of x. Greater integer of x is also called integral part of x.

$y = f(x) = [x]$

Domain : $\boldsymbol{R}$; Range : $\boldsymbol{I}$

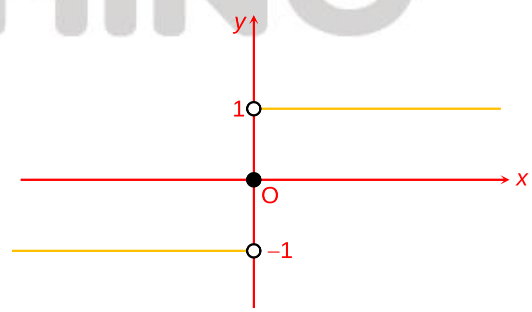


➢ **Signum function:** The function is defined as

$y = f(x) = \text{sgn}(x)$

$$\text{sgn}(x) = \begin{cases} \dfrac{| x |}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\text{or} \quad \text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

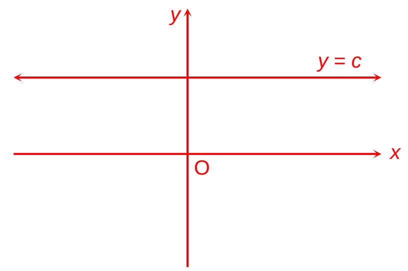


Domain : $\boldsymbol{R}$; Range $\rightarrow \{-1, 0, 1\}$

- **Rational algebraic function:** A function of the form $f(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function.

  The domain of a rational function $\frac{p(x)}{q(x)}$ is the set of all real numbers except points where $q(x) = 0$.

- **Constant function:** The function defined as $f : \mathbf{R} \to \{c\}$ where $f(x) = c$



9 ALGEBRAIC OPERATIONS ON FUNCTIONS

Let us consider two functions.

$f : D_1 \to \mathbf{R}$ and $g : D_2 \to \mathbf{R}$. We describe functions $f + g$, $f - g$, $f.g$ and f/g as follows:

- $f + g : D \to \mathbf{R}$ is a function defined by
  $(f + g)\, x = f(x) + g(x)$, where $D = D_1 \cap D_2$
- $f - g : D \to \mathbf{R}$ is a function defined by
  $(f - g)\, x = f(x) - g(x)$, where $D = D_1 \cap D_2$
- $f \,.\, g : D \to \mathbf{R}$ is a function defined by
  $(f \,.\, g)\, x = f(x) \,.\, g(x)$, where $D = D_1 \cap D_2$
- $f / g : D \to \mathbf{R}$ is a function defined by
  $(f / g)\, x = \frac{f(x)}{g(x)}$, where $D = D_1 \cap \{x \in D_2 : g(x) \neq 0\}$
- $(\alpha f)(x) = \alpha f(x)$, $x \in D_1$ and α is any real number.

10 TYPES OF FUNCTIONS

We have seen that f is a function from A to B, if each element of A has image in B and no element of A has more than one images in B.

But for a function *f* from *A* to *B* following possibilities are there

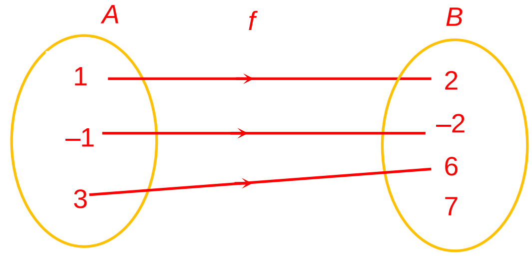
- Distinct elements of A have distinct images in B.
- More then one element of A may have same image in B.
- Each element of B is the image of some element of A.
- There may be some elements in B which are not the images of any element of A.

Because of the above mentioned possibilities, we have the following types of functions:

10.1 One-one or injective map

A map $f : A \to B$ is said to be one-one or injective if each and every element of set A has distinct images in set B.

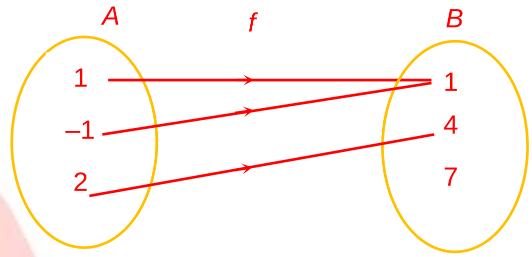
The map $f : A\{-1, 1, 3\} \to B\{-2, 2, 6, 7\}$ given by $f(x) = 2x$ is a one-one map.



10.2 Many one map:

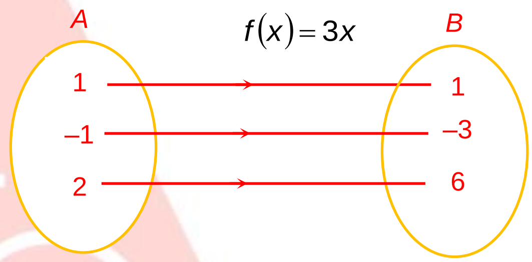
A map $f : A\{-1, 1, 2\} \to B\{1, 4, 7\}$ is said to be many one if and only if it is not one-one.

The map $f : A \to B$ given by $f(x) = x^2$ is a many-one map.



10.3 Onto map or surjective map:

A map $f : A \to B$ is said to be onto map or surjective map if and only if each element of B is the image of some element of A i.e. if and only if for every $y \in B$ there exists some $x \in A$ such that $y = f(x)$.



Thus f is onto iff $f(A) = B$ i.e. range of f = co-domain of f.

A map $f : A\{1, -1, 2\} \to B\{1, -3, 6\}$ given by $f(x) = 3x$ is an onto map.

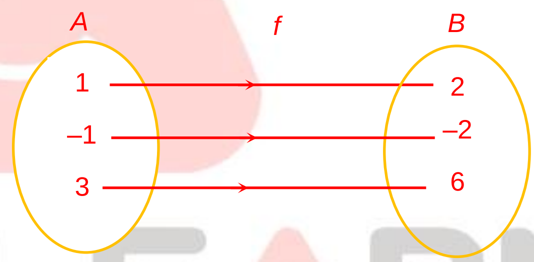
Note: Functions which are not onto, are into.

10.4 One-one onto map or bijective map:

A map $f : A \to B$ is said to be one-one onto or bijective if and only if it is both one-one and onto i.e., if

(i) distinct element of A have distinct images in B.

(ii) each element of B is the image of some element of A.



The map $f : A\{1, -1, 3\} \to B\{2, -2, 6\}$ given by $f(x) = 2x$ is a one-one onto map.

- A one-one onto function is also called a one-to-one correspondence or one-one correspondence.
- Let $f : A \to B$ be a function from finite set A to finite set B. Then
  1. f is one-one $\Rightarrow$ $n(A) \le n(B)$
  2. f is onto $\Rightarrow$ $n(B) \le n(A)$
  3. f is one-one onto $\Rightarrow$ $n(A) = n(B)$

11 COMPOSITION OF FUNCTIONS

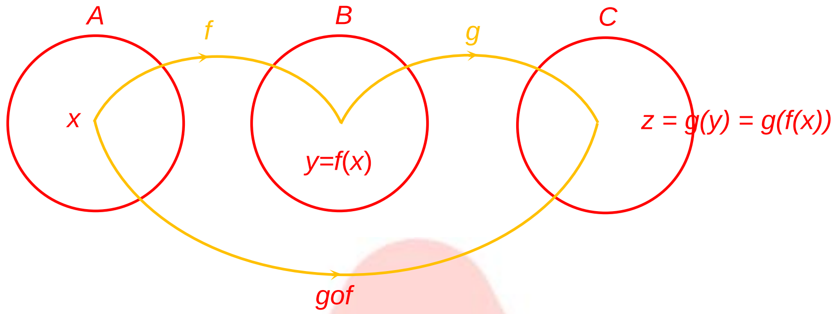
Let A, B, C be three non-empty sets, f be a function from A to B and g be a function from B to C. The question arises : can we combine these two functions to get a new function? Yes! The most natural way of doing this is to send every element $x \in A$ in two stages to an element of C; first by applying f to x and then by applying g to the resulting element $f(x)$ of B.

DEFINITION

Let $f : A \to B$ and $g : B \to C$ be any two mappings. Then f maps an element $x \in A$ to an element $f(x) = y \in B$ and this y is mapped by g to an element $z \in C$. Thus $z = g(y) = g(f(x))$

Thus we have a rule, which associates with each $x \in A$, a unique element $z = g(f(x))$ of C. This rule is therefore a mapping from A to C. We denote this mapping by gof (read as 'g composition f') and call it the composite mapping of f and g.

$$(gof)(x) = g(f(x)) \ \forall \ x \in A$$



The composition of two functions is also called the resultant of two functions or the function of a function.

Observe that the order of events occur from right to left i.e. gof reads composite of f and g and it means that we have to first apply f and then follow it up with g.

Note that for the composite function gof to exist, it is essential that range of f must be a subset of domain of g.

(i) Dom. (gof) = {x : $x \in$ domain (f), $f(x) \in$ domain (g)}

(ii) If gof is defined then it is not necessary that fog is defined.

Illustration 6

Question: **Let $f = \{(1, 2), (2, 3), (4, 5)\}$ and $g = \{(2, 3), (3, 5), (5, 2)\}$. Check whether *gof* and *fog* is defined, also find the range of *gof*.**

Solution: $\because \ f = \{(1, 2), (2, 3), (4, 5)\}, \ g = \{(2, 3), (3, 5), (5, 2)\}$

Then dom . f = {1, 2, 4}; Range f = {2, 3, 5}; dom.g = {2, 3, 5}; Range g = {3, 5, 2}

since dom. g = Range f, $\therefore$ gof is defined

But dom . $f \neq$ Range g, $\therefore$ fog is not defined.

Also in this particular example, dom. (gof) = dom . f = {1, 2, 4}

$$(gof)(1) = g[f(1)] = g(2) = 3$$
$$(gof)(2) = g[f(2)] = g(3) = 5$$
$$(gof)(4) = g[f(4)] = g(5) = 2$$

Hence range of gof is {2, 3, 5}.

12 INVERSE FUNCTION

Let f be one-one and onto map from A to B. Since f is onto, therefore $\forall \ y \in B$ there exist $x \in A$ such that $f(x) = y$ and since f is one-one therefore this element x is unique. Thus we can define a map, say g from B onto A such that $g(y) = x$. This map g is called inverse map of f and is denoted by f^{-1}.

Thus $f^{-1} : B \to A$ such that $f^{-1}(y) = x$ iff $f(x) = y$

- **How to find the inverse of a given function?**

In order to find the inverse of the function $f(x)$, let $y = f(x)$

From this express x in terms of y. This value of x in terms of y will be $f^{-1}(y)$. Now put x in place of y in $f^{-1}(y)$ to get $f^{-1}(x)$.

Note: f^{-1} exists if and only if f is one-one onto.

Let $y = f(x) \Rightarrow y = 2x \Rightarrow x = \frac{y}{2} \Rightarrow f^{-1}(y) = \frac{y}{2} \Rightarrow f^{-1}(x) = \frac{x}{2}$