Sequence & Series

Informally, a sequence is an ordered list of objects, but in this lesson the objects will usually be numbers.

e.g., 2, 0, –2, 2, 0, –2, 2, 0, –2, …………………;
1, 2, 4, 8, 16, ……………………;
$x, x - 3, x - 6, x - 9$,………, *etc*.

You can observe that there is a pattern in each of these list of numbers/expressions. This pattern may be easy to observe and sometimes difficult. But with the help of this pattern we can tell the number at any position in the sequence.

It is obvious that there is no limit to the kinds of patterns we can form and thus infinite types of sequences can be generated. In this lesson, we will focus our attention to some basic sequences. Before that let us define certain terms.

Sequence : A set of numbers arranged in some definite order and according to some definite rule is called a sequence.
Alternatively, a real sequence is a function whose domain is the set of natural numbers N or a subset of N, and range is a set of real numbers.

Series : Terms of a sequence when added form a series.

Progression : When terms of a sequence are written under specific conditions, then the sequence is called a progression.

A progression is represented as $t_1, t_2,\ldots\ldots,t_n,\ldots\ldots$ or $a_1, a_2,\ldots\ldots,a_n,\ldots\ldots$ where t_1 (or a_1) means first term, and t_n (or a_n) means n^{th} term. t_k(or a_k) is called the general term of the progression. The number of terms can be finite or infinite.

$S_n = t_1 + t_2 + \ldots\ldots + t_n$, denotes the sum of corresponding series upto n terms.

Clearly, $t_n = S_n - S_{n-1}$.

1 ARITHMETIC PROGRESSION

A succession of terms is said to be in Arithmetic Progression if the difference between any term and its preceding term is independent of the position of the term.

Arithmetic Progression is abbreviated as A.P.

e.g., 5, 9, 13, 17, 21, ...;
$t, t - 5, t - 10, t - 15$,...........; *etc*.

We define,

Common difference = value of a term – value of the preceding term.

In the above examples, common difference (c.d.) is 4 and –5 respectively.

The first term is usually denoted by 'a' and the common difference by 'd'. So, an A.P. can be written as: $a, a + d, a + 2d, a + 3d$, ...

If follows that the k^{th} term of an A.P., *i.e.*, t_k (or a_k) is given by : $t_k = a + (k - 1)d$ (General term)

If an A.P. consists of n terms, its n^{th} term is called its last term denoted by l and is given by: $l = t_n = a + (n - 1)d$.

Illustration 1

Question: **If p times the q^{th} term of an A.P. is equal to q times its p^{th} term, show that $t_{p-q} : t_{p+q} = p - q : p + q$.**

Solution: Let us proceed backwards.

We require $t_{p-q} : t_{p+q}$ in terms of p and q only.

If we assume the first term as 'a' and common difference as 'd', then we must obtain a relation between them to get the required ratio in that form.

Now, $\dfrac{t_{p-q}}{t_{p+q}} = \dfrac{a+(p-q-1)d}{a+(p+q-1)d}$

But from the given condition,

$pt_q = qt_p$

$\Rightarrow$ $p[a + (q - 1)d] = q[a + (p - 1)d]$

$\Rightarrow$ $(p - q)\, a = (pq - q - pq + p)d$

$\Rightarrow$ $a = d$ $\qquad (\because p \neq q)$

Hence, $\dfrac{t_{p-q}}{t_{p+q}} = \dfrac{a+(p-q-1)a}{a+(p+q-1)a} = \dfrac{p-q}{p+q}$.

1.1 SUM OF AN ARITHMETIC PROGRESSION

If an A.P. contains n terms, the corresponding series can be summed.
Sum to n terms of an A.P. is given by

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right] = \frac{n}{2}(a+l)$$

Proof :

Consider the following addition:

$S = 2 + 8 + 14 + 20 + 26 + 32 + 38$

$S = 38 + 32 + 26 + 20 + 14 + 8 + 2$

Adding $\quad 2S = (40 + 40 + 40 + 40 + 40 + 40 + 40)$

i.e., $\quad 2S = 7 \times 40$

$S = \dfrac{7}{2} \times 40 = \dfrac{7}{2}$ (first term + last term)

$= \dfrac{n}{2}$ (first term + last term)

where, n is the number of terms of the A.P.

last term $= t_n = a + (n - 1)d$

$\therefore \qquad S = \dfrac{n}{2}[a+t_n] = \dfrac{n}{2}[a+a+(n-1)d] = \dfrac{n}{2}[2a+(n-1)d]$

The above formula gives us the sum of n terms of an A.P.

Illustration 2

Question: **The sum of the first 13 terms of an A.P. is 21 and the sum of first 21 terms is 13. Find the sum of first 34 terms.**

Solution: Given $S_{13} = 21$ and $S_{21} = 13$

i.e., $\dfrac{13}{2}[2a+(13-1)d] = 21$

$\Rightarrow \quad a + 6d = \dfrac{21}{13}$ ...(i)

and $\dfrac{21}{2}[a+(21-1)d] = 13$

$\Rightarrow \quad a + 10d = \dfrac{13}{21}$ ...(ii)

From equation (i) & (ii),

$a = \dfrac{849}{273}$ and $d = -\dfrac{68}{273}$

$\therefore S_{34} = \dfrac{34}{2}[2a+(34-1)d] = -34$

Illustration 3

Question: **Find the arithmetic progression, consisting of 10 terms if the sum of the terms occupying the even places is equal to 15 and the sum of those occupying the odd places is equal to**

$12\frac{1}{2}$.

Solution: Let the successive terms of the A.P. be $a_1, a_2, \ldots, a_9, a_{10}$.
By hypothesis $a_2 + a_4 + a_6 + a_8 + a_{10} = 15$
i.e., $(a + d) + (a + 3d) + \ldots + (a + 9d) = 15$
i.e., $5a + 25d = 15$ … (i)

and $a_1 + a_3 + a_5 + a_7 + a_9 = 12\frac{1}{2}$

$5a + 20d = 12\frac{1}{2}$ … (ii)

From (i) and (ii), we get $5d = 2\frac{1}{2}$ or $d = \frac{1}{2}$ and $a = \frac{1}{2}$

Hence the A.P. is $\frac{1}{2}, 1, 1\frac{1}{2}, 2, \ldots$.

1.2 IMPORTANT FACTS

(i) If a constant is added to (or subtracted from) every term of an A.P., the resulting sequence will also be an A.P. with the same common difference;

(ii) If every term of an A.P. is multiplied by (or divided by) a fixed constant, the resulting sequence will also be an A.P.

(iii) It is convenient in problems dealing with only three terms to take the three terms in A.P. as $a - d, a, a + d$, with a as the middle term and d as the common difference.

(iv) Four terms in A.P. can be taken as $a - 3d, a - d, a + d, a + 3d$. Note that $2d$ is the common difference.

Illustration 4

Question: **Find four numbers in A.P. such that their sum is *50* and greatest of them is *4* times the least.**

Solution : Let the four numbers in *A.P.* be $\alpha - 3\beta, \alpha - \beta, \alpha + \beta, \alpha + 3\beta$.
Given, $\alpha - 3\beta + \alpha - \beta + \alpha + \beta + \alpha + 3\beta = 50$

or $4\alpha = 50 \quad \therefore \alpha = \frac{25}{2}$

and $\alpha + 3\beta = 4(\alpha - 3\beta)$ or $3\alpha = 15\beta$

or $\beta = \frac{\alpha}{5} = \frac{25}{5 \times 2} = \frac{5}{2}$

Hence, the four numbers are 5, 10, 15, 20.

Illustration 5

Question: **If the numbers $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ form an A.P., then prove that a^2, b^2, c^2 are also in A.P.**

Solution: $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$\Rightarrow (c + a)(a + b), (b + c)(a + b), (b + c)(c + a)$ are in A.P.
[by multiplying by $(b + c)(c + a)(a + b)$]
$\Rightarrow (a^2 + ab + bc + ca), (b^2 + ba + bc + ac), (c^2 + ca + cb + ab)$ are in A.P.
$\Rightarrow a^2, b^2, c^2$ are in A.P. (Subtracting ab + bc + ca from all the three terms)

2 GEOMETRIC PROGRESSION

A succession of terms is said to be in Geometric Progression if the ratio of a term to its preceding term is independent of the position of the term. Geometric Progression is abbreviated as G.P.

e.g., 3, 15, 75, 375,............;
$t, t/3, t/9, t/27$, ...; *etc.*

We define

Common ratio $= \dfrac{\text{value of a term}}{\text{value of its preceding term}}$

In the above examples, common ratio (c.r.) is 5 and 1/3 respectively.
The first term is usually denoted by 'a' and the common ratio by 'r'. So, a G.P. can be written as
a, ar, ar^2, ar^3, ...
So, k^{th} term of a G.P. is given by
$t_k = ar^{k-1}$ (General term)
For a G.P. consisting of n terms, the last term is
$l = t_n = ar^{n-1}$

Illustration 6

Question: **If the fifth term of a G.P. is 81 and second term is 24, find the G.P.**

Solution: Here $t_5 = 81$ and $t_2 = 24$

$\therefore t_n = ar^{n-1}$ $\quad\therefore\quad 81 = ar^4$ ...(i)

and $24 = ar^{2-1}$ or $24 = ar$ ...(ii)

dividing (i) by (ii), we get

$$\frac{ar^4}{ar} = \frac{81}{24} = \frac{27}{8} \quad \text{or} \quad r^3 = \left(\frac{3}{2}\right)^3 \quad \therefore \quad r = \frac{3}{2}$$

Putting the value of r in (ii), we get $a = \dfrac{24}{r} = 24.\dfrac{2}{3} = 16$

Hence the required G.P. is 16, 24, 36, 54,...... .

2.1 SUM OF TERMS IN G.P.

Sum to n terms of a G.P. is given by

$$S_n = \frac{a(1-r^n)}{1-r} \text{ provided } r \neq 1$$

The sum formula shows that if the magnitude of r is less than 1, r^n decreases as n increases. If n becomes very large, *i.e.*, as $n \to \infty$, $r^n \to 0$. So, if the number of terms is very large, we can get the sum of a G.P. to infinite terms, as a finite quantity.

$$\therefore \quad S_\infty = \frac{a}{1-r} \quad \text{if} \quad |r| < 1$$

Proof :

Let $S_n = a + ar + ar^2 + ... + ar^{n-1}$ ...(i)

Consider two cases : $r = 1$, $r \neq 1$

If $r = 1$, $S_n = \underbrace{a + a + a + ...+ a}_{n\text{ times}} = na$

If $r \neq 1$, multiplying (i) by r, we have

$rS_n = ar + ar^2 + ar^3 + ... + ar^n$ ...(ii)

Subtracting (ii) from (i), we get

$$(1-r)S_n = a - ar^n \quad \Rightarrow \quad S_n = \frac{a(1-r^n)}{1-r}$$

For convenience, we write $S_n = \dfrac{a(r^n-1)}{r-1}$ if $|r| > 1$.

$$\therefore \quad S_n = \begin{cases} \dfrac{a(1-r^n)}{1-r} & \text{if} \quad -1 \leq r < 1 \\ na & \text{if} \quad r = 1 \\ \dfrac{a(r^n-1)}{r-1} & \text{if} \quad |r| > 1 \; i.e., r < -1 \text{ or } r > 1 \end{cases}$$

What happens if $r = -1$?

In this case the G.P. becomes $a, -a, a, -a$, ...

So the sum depends on the number of terms.

$S_n = 0$ if $n = 2, 4, 6$, ...*i.e.,* when n is even
and $S_n = a$ if $n = 1, 3, 5$, ...*i.e.,* when n is odd.

The formula $S_n = \frac{a(1-r^n)}{1-r}$ holds in this case.

Let us apply the S_n formula for the series with

$a = 10$ and $r = -\frac{1}{2}$

$$\therefore S_n = \frac{a(1-r^n)}{1-r} = \frac{(10)\left[1-\left(-\frac{1}{2}\right)^n\right]}{1-\left(-\frac{1}{2}\right)}$$

The quantity $\left(-\frac{1}{2}\right)^n$ becomes nearly zero if n is very large.

$$\therefore S_\infty = \frac{10(1-0)}{1+\frac{1}{2}} = \frac{20}{3}$$

So, the sum of infinite number of terms is finite.
Therefore, *a* G.P. can be summed till infinity if $|r| < |$.
(because, as $n \to \infty,\ r^n \to 0$)

Hence, $S_\infty = \frac{a}{1-r}$, if $|r| < 1$ *i.e.*, $-1 < r < 1$.

Illustration 7

Question: **Find the least value of n for which $1 + 3 + 3^2 + ...+ 3^{n-1} > 1000$.**

Solution : Given, $1 + 3 + 3^2 + ...+ 3^{n-1} > 1000$

or $1\left(\frac{3^n - 1}{3-1}\right) > 1000$

or $3^n - 1 > 2000$ or $3^n > 2001$ ...(i)

Now $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, $3^6 = 729$, $3^7 = 2187$, clearly $2187 > 2001$,

$\therefore$ least value of n for which $3^n > 2001$ is 7.

Illustration 8

Question: **If s is the sum to infinity of a G.P. whose first term is a, then prove that the sum of the first n terms is $\left[1-\left(1-\frac{a}{s}\right)^n\right]s$.**

Solution: If r be the common ratio, we have $s = \frac{a}{1-r}$

$\therefore r = 1 - \frac{a}{s}$

$$S_n = a\left[\frac{1-r^n}{1-r}\right] = a\left[\frac{1-\left(1-\frac{a}{s}\right)^n}{1-\left(1-\frac{a}{s}\right)}\right] = \frac{a\left[1-\left(1-\frac{a}{s}\right)^n\right]}{\frac{a}{s}}$$

$$S_n = \left[1-\left(1-\frac{a}{s}\right)^n\right]s$$

2.2 SOME RESULTS FOR G.P.

1. If a, b, c are in G.P., then for $k\ (\neq 0)$
   ak, bk, ck are also in G.P. $a/k, b/k, c/k$ are also in G.P.
2. If $a_1, a_2, a_3,$ …………….. and b_1, b_2, b_3 ………………… are two G.P.'s, then the sequence $a_1b_1, a_2b_2, a_3b_3;$ ……………..is also in G.P.
3. Three terms in a G.P. are taken as $\frac{a}{r}, a, ar.$
4. Four terms in a G.P., are taken as $\frac{a}{r^3}, \frac{a}{r}, a, ar^3.$
5. If a, b, c and d are in G.P., then $\frac{a}{b}=\frac{b}{c}=\frac{c}{d} \Rightarrow b^2=ac,\ c^2=bd,\ ad=bc.$
6. Given a_1, a_2, a_3, ……..$(a_i > 0\ \forall i)$ are in G.P. then $\log a_1$, $\log a_2$, $\log a_3$,……… are in A.P. and vice versa.

Illustration 9

Question: **If the product of three number in G.P. be 216 and their sum is 19, find the numbers.**

Solution : Let the three numbers be $\frac{a}{r}, a, ar$

Given, $\frac{a}{r}.a.ar = 216$ or $a^3 = 6^3$ $\therefore a = 6$

Also $\frac{a}{r}+a+ar=19$ $\therefore \frac{6}{r}+6+6r=19$

or $6r^2 - 13r + 6 = 0$ or $6r^2 - 9r - 4r + 6 = 0$

or $3r(2r-3) - 2(2r-3) = 0$ or $(3r-2)(2r-3) = 0$

$\therefore r = \frac{2}{3}, \frac{3}{2}$

When $r = \frac{2}{3}$, numbers will be 9, 6 and 4

when $r = \frac{3}{2}$, numbers will be 4, 6, 9. Hence the numbers are 4, 6, 9.

3 HARMONIC PROGRESSION

The sequence $a_1, a_2, a_3, \ldots, a_n$ is said to be a Harmonic Progression (H.P.) if the sequence formed by the reciprocals, $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}$ is an A.P.

Common difference of this A.P. $= \frac{1}{a_2} - \frac{1}{a_1} = \frac{a_1 - a_2}{a_2 a_1} = d$

and t_n of this A.P. $= \frac{1}{a_1} + (n-1)d$

Hence the n^{th} term of the H.P. $= \frac{1}{a+(n-1)d}$, where $a = \frac{1}{a_1}$

Illustration 10

Question: **If a, b, c are in H.P., then show that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.**

Solution: a, b, c are in H.P. $\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in A.P. (Multiplying by $a + b + c$)

$\Rightarrow 1 + \frac{b+c}{a}, 1 + \frac{c+a}{b}, 1 + \frac{a+b}{c}$ are in A.P.

$\Rightarrow \frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P. (Subtracting 1 from each)

$\Rightarrow \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

Illustration 11

Question: **If the 7th term of a H.P. is 8 and the 8th term is 7, then find its 15th term.**

Solution: t_7 of H.P. = 8 and t_8 of H.P. = 7

$\Rightarrow \frac{1}{a+6d} = 8, \frac{1}{a+7d} = 7$

Solving, $d = \frac{1}{56} = a$ $\quad \therefore T_{15}$ of H.P. is $\frac{1}{a+14d} = \frac{56}{15}$