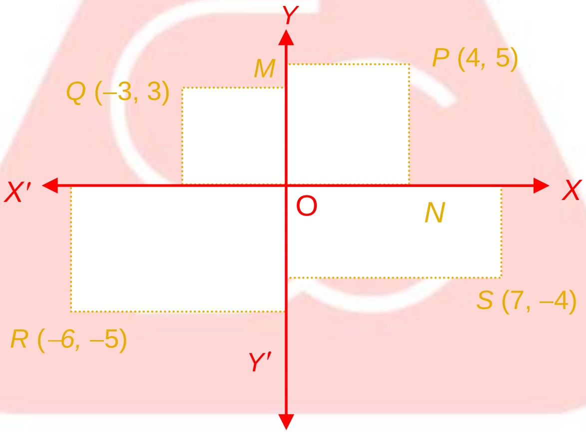
Points & Straight Lines

1 PRELIMINARIES

The system of Geometry in which a point is specified by means of an ordered number-pair is known as Coordinate Geometry. It enables us to solve geometrical problems by algebraic methods.

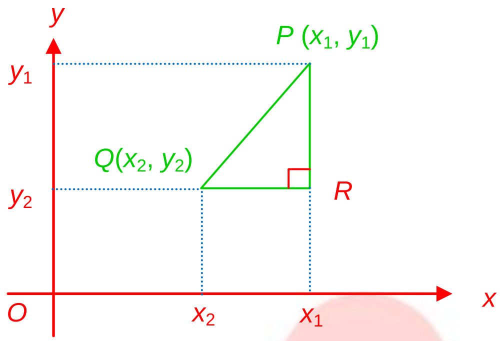
1.1 COORDINATES OF POINT

Let *P* be a point in a plane; draw two perpendicular lines $X'OX, Y'OY$ in the plane, and draw *PM, PN* parallel to *OX, OY* respectively. $X'OX, Y'OY$ are known as the coordinate axes. *MP* is the *x*-coordinate, called the abscissa, and *NP* is the *y*-coordinate called the ordinate of *P*. These two distances *MP* and *NP* together fix the position of *P* in the plane.



If $MP = 4$, $NP = 5$, then the position of *P* is denoted by (4, 5) the *x*-coordinate being written first. *Q* is the point (–3, 3), for the *x*-coordinate, is measured to the left of the *y*-axis, and is negative in sign; *R* is the point (–6, –5) for both coordinates are negative in sign. Similarly *S* is the point (7, –4). It should be observed that *O*, the origin is the point (0, 0) and the *y*-coordinate of any point on the *x*-axis is zero while the *x*-coordinate of any point on the *y-axis* is zero.

Every point in the plane of the axes of coordinates is represented correspondingly by an ordered pair *(x, y)* the first one, namely, *x,* representing the *x*-coordinate of the point and the second one, namely *y,* representing the *y*-coordinate of the point. Conversely to every ordered pair there corresponds only one point. Thus this method of representation of points in a plane by an ordered pair of real number is, what is usually called one-one and onto by which we mean that for every point there is only one ordered pair of real numbers and for every ordered pair of real numbers there is only one point, and neither any point nor any ordered pairs is left out without its associated ordered pair of real numbers or a point as the case may be.



Let the coordinates of the given points P and Q be (x_1, y_1) and (x_2, y_2) respectively.

From the Figure

$$PQ = \sqrt{QR^2 + PR^2}$$

i.e., $$\mathbf{PQ} = \sqrt{(\mathbf{x_1} - \mathbf{x_2})^2 + (\mathbf{y_1} - \mathbf{y_2})^2}$$

Illustration 1

Question: **Show that the points (2, 3) (1, 5) (–2, 0) and (–1, –2) are vertices of a parallelogram.**

Solution: Let the points be denoted by A, B, C, D in order

$AB^2 = (2-1)^2 + (3-5)^2$

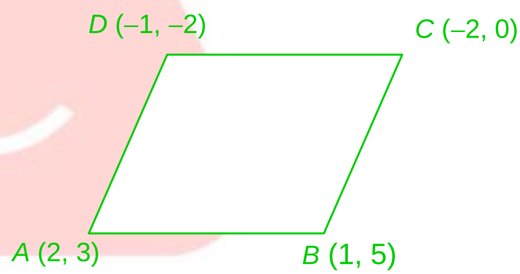
= 1 + 4 = 5

$BC^2 = (1+2)^2 + (5-0)^2$

= 9 + 25 = 34

$CD^2 = (-2+1)^2 + (0+2)^2$

=1 + 4 = 5



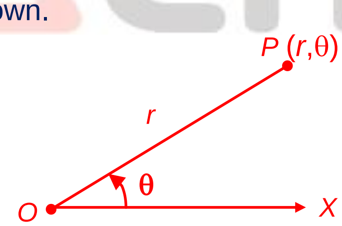
$DA^2 = (-1-2)^2 + (-2-3)^2$

= 9 + 25 = 34

Since the opposite sides are equal, the figure drawn is a parallelogram.

1.3 POLAR COORDINATES

Let OX be a given line and P be any point in a given plane. Let θ be the angle through which a line rotates in moving from OX to the position OP. Then, if r is the length of OP, the position of P is known when r and θ are known.



r is called the radius vector, θ the vectorial angle and (r, θ) is the polar coordinates of the point P, which is briefly referred to as the point (r, θ). OX is called the initial line and O the pole.

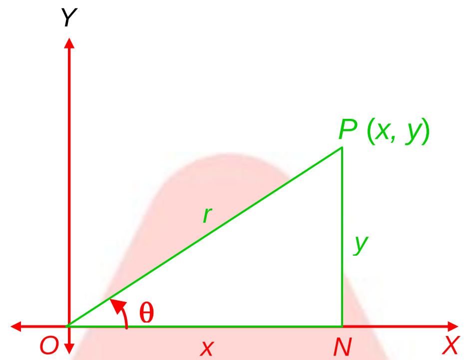
1.4 RELATION BETWEEN CARTESIAN AND POLAR COORDINATES

Let P be the point (x, y) with reference to rectangular axes OX, OY and the point (r, θ) with reference to the pole O and initial line OX. i.e., the X-axis.

$\therefore$ we have $x = r\cos\theta$; $y = r\sin\theta$

Hence $\tan\theta = \dfrac{y}{x}$ and $x^2 + y^2 = r^2$

These relations enable us to change from one system of coordinates to the other.

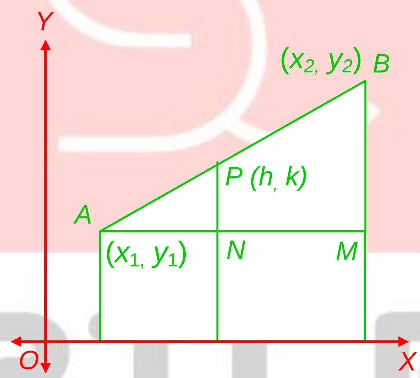


2 SECTION FORMULA

Let *A, B* are the points (x_1, y_1), (x_2, y_2), here we will find the coordinates of the point *P* on *AB* such that *AP : PB = l : m.*

Let the coordinates of *P* be (*h, k*)

In the Figure, triangles *ANP* and *AMB* are similar.



$\therefore \quad \dfrac{PN}{BM} = \dfrac{AP}{AB} = \dfrac{l}{l+m}$, since $\dfrac{AP}{PB} = \dfrac{l}{m}$

i.e., $\dfrac{k-y_1}{y_2-y_1} = \dfrac{l}{l+m}$

$\therefore \quad (l+m)k = (l+m)y_1 + l(y_2 - y_1)$

$= ly_2 + my_1$

$\therefore\ k = \dfrac{ly_2 + my_1}{l+m}$

similarly, $h = \dfrac{lx_2 + mx_1}{l+m}$

Note: Putting *l = m,* we find that the mid-point of the line joining the points $(x_1, y_1), (x_2, y_2)$, i.e. $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

Note : When *P* lies outside *AB* i.e., external to *AB* such that *AP : BP = l : m.*

We have h = $\dfrac{lx_2 - mx_1}{l-m}, k = \dfrac{ly_2 - my_1}{l-m}$

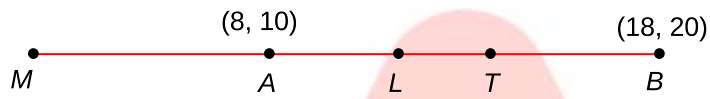
It has been assumed that *l* and *m* are positive numbers; if, however, we take *AP : BP* to be positive in the former case and negative in the latter case, we have in both cases the formula

$$h = \frac{lx_2 + mx_1}{l + m}, k = \frac{ly_2 + my_1}{l + m}$$

Illustration 2

Question: **Find the coordinates of the points *L, M* which divide internally and externally in the ratio *2 : 3* the line joining the points *A* (8, 10), *B* (18, 20), and show that *TL . TM = TB*^{2}, where *T* is the midpoint of *AB*.**

Solution:



Since *AL : LB* = 2 : 3, the coordinates of *L* are $\left(\frac{2x18 + 3x8}{2+3}, \frac{2x20 + 3x10}{2+3}\right)$

i.e., (12, 14)

Since *M* divides *AB* externally in the ratio 2 : 3, abscissa = $\frac{2x18 - 3x8}{2-3} = -12$

Ordinate = $\frac{2x20 - 3x10}{2-3} = -10$

Since *T* is the mid-point of *AB*, the coordinates of *T* are $\left(\frac{8+18}{2}, \frac{10+20}{2}\right)$ *i.e.,* (13, 15)

$T(13,15), L(12,14), M(-12,-10), B(18,20)$

$TL = \sqrt{(13-12)^2 + (15-14)^2} = \sqrt{2}$

$TM = \sqrt{(13+12)^2 + (15+10)^2} = 25\sqrt{2}$

$\therefore$ *TL . TM* = 50

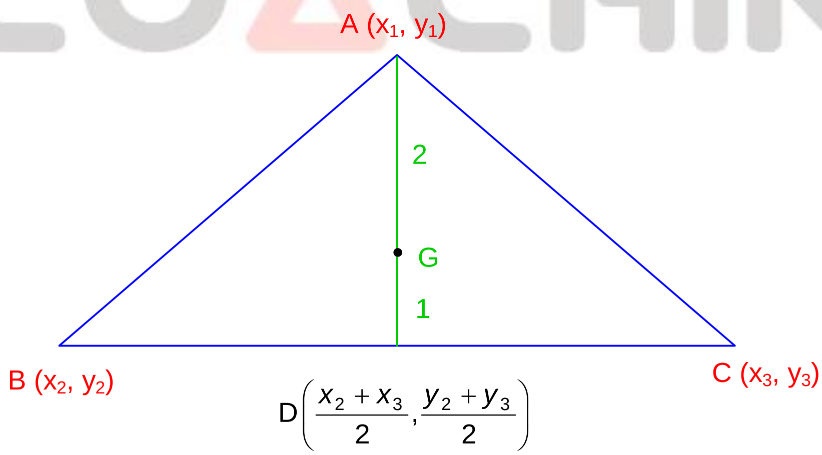
$TB^2 = (13 - 18)^2 + (15 - 20)^2 = 50$

$\therefore$ *TL . TM* = TB^2

Illustration 3

Question: **_A, B, C_ are the points (x_1, y_1), (x_2, y_2), (x_3, y_3). *D* is the mid point of *BC, and G* divides *AD* in the ratio 2 : 1, find the coordinates of *G* and show that the medians of triangle *ABC* are concurrent.**

Solution: Since *BD = DC, D* is the point $\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}$



Since *AG : GD* = 2 : 1, the coordinates of *G* are

$$\left(\frac{1.x_1 + 2\left(\frac{x_2 + x_3}{2}\right)}{1+2}, \frac{1.y_1 + 2\left(\frac{y_2 + y_3}{2}\right)}{1+2}\right)$$

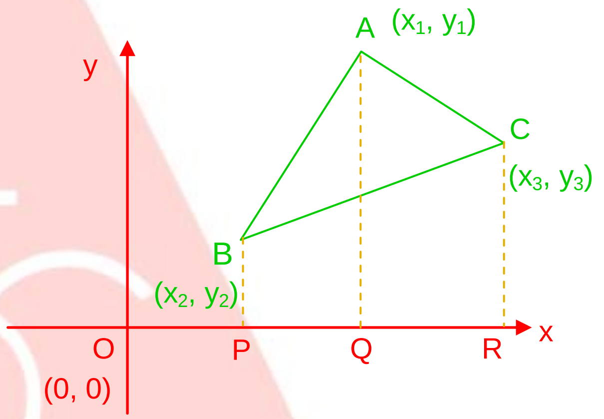
i.e., $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

The symmetry of this result in x_1, x_2, x_3 and in y_1, y_2, y_3 shows that *G* lies on each median. Hence the medians of triangle *ABC* are concurrent.

3 AREA OF A TRIANGLE

Let the vertices of the triangle be

$A(x_1, y_1), \; B(x_2, y_2), \; C(x_3, y_3)$



We have considered the vertices of the triangle in the first quadrant for the ease of calculations. Drop perpendiculars form A, B and C on the x-axis.

Area of ΔABC = Area of trapezium ABPQ

+ Area of trapezium AQRC

– Area of trapezium BCRP

$$= \frac{1}{2}(BP + AQ)(PQ) + \frac{1}{2}(AQ + CR)(QR) - \frac{1}{2}(BP + CR)(PR)$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

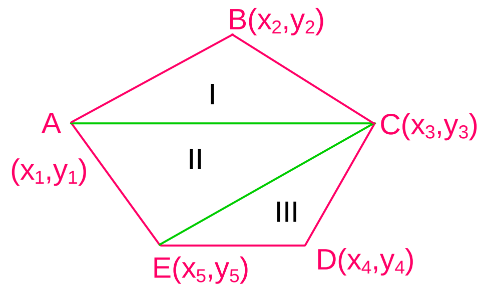
expressed in the determinant form,

$$\Rightarrow \textbf{Δ = absolute value of } \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

The area of a triangle is considered positive when its vertices are named in counter-clockwise order, and negative when named in clockwise order.

If three points are collinear, we have the area of the Δ = 0.

Similarly to calculate the area of pentagon whose vertices are given, we can divide the pentagon into a number of triangles and then sum up the areas. e.q.



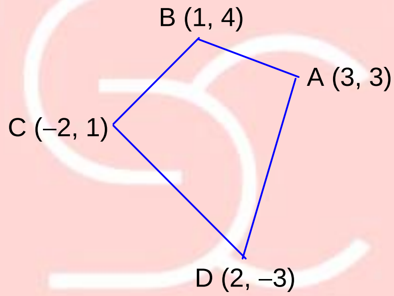
Area of pentagon *ABCDE* = Area of I + Area of II + Area of III

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_5 & y_5 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \\ x_5 & y_5 & 1 \end{vmatrix}$$

Illustration 4

Question: **Find the area of the quadrilateral with vertices (3, 3), (1, 4), (–2, 1), (2, –3).**

Solution: Let the given vertices be *A, B, C, D* respectively.



Join *AC*

Area of *ABCD* = area of triangle *ABC* + area of triangle *ACD*

$$= \frac{1}{2}[3(4-1)+1(1-3)-2(3-4)] + \frac{1}{2}[3(1+3)-2(-3-3)+2(3-1)]$$

$$= \frac{1}{2}(9-2+2+12+12+4) = \frac{37}{2}.$$

4 EQUATION OF LOCUS OF A POINT

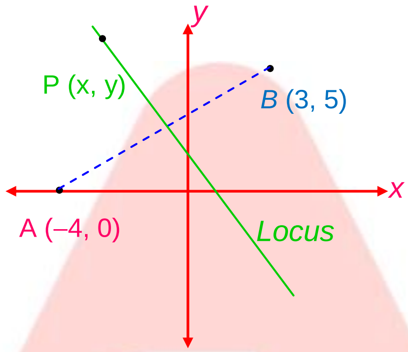
The locus of a point, as it moves in accordance with a given geometrical condition, is the path traced out by the moving point. In geometry we mean, by locus, the curve itself. We say that the locus is a straight line (or) the locus is a circle. On the contrary in coordinate geometry the moving point can be represented by the ordered pair (*x, y*). The geometrical condition imposed on the moving point gets transformed into an algebraic relation connecting (*x, y*). This relation, between *x* and *y* (the coordinates of the moving point), is called the equation to the curve described by the points.

For example, if $P(x, y)$ is any point equidistant from $A(-4, 0)$ and $B(3, 5)$, we have $PA^2 = PB^2$ and this can be expressed in terms of coordinates of A, P and B as

$$(x+4)^2 + y^2 = (x-3)^2 + (y-5)^2$$

i.e., $14x + 10y - 18 = 0$

i.e., $7x + 5y - 9 = 0$



Moreover, if $P(x, y)$ is any point whose coordinates satisfy the equation $7x + 5y - 9 = 0$, We have $PA^2 = PB^2$

i.e., P is equidistant from A and B. Hence the relation $7x + 5y - 9 = 0$, is called the equation of the locus of points equidistant from A and B.

It is said that the equation $7x + 5y - 9 = 0$, represents the perpendicular bisector of AB.

Coordinate Geometry is based on the concept of locus of a point. Large number of problems will involve the idea of the locus of a point.

To find locus of a point *P*, proceed as follows:

1. Assume co-ordinate of P as (h, k); h and k are unknown quantities.
2. Translate the given problem into equations, which will contain coordinates of P, some given quantities and some unknown quantities.
3. Suppose two unknown quantities are introduced in the above method. Then one must get three equations (or more) in step 2.
4. Eliminate unknowns to get an equation which should be generalized by replacing h by x, k by y, to get the required locus.

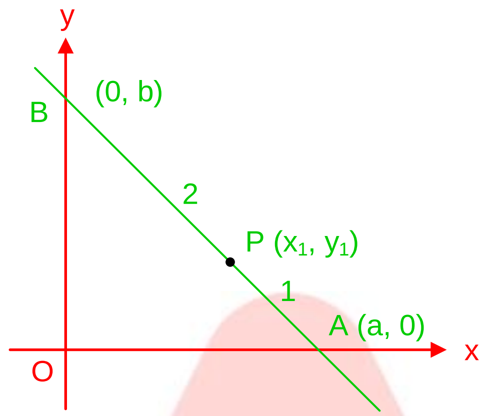
Illustration 5

Question: **A straight line segment of length *'p'* moves with its ends on two mutually perpendicular lines. Find the locus of the point which divides the line in the ratio 1 : 2.**

Solution: Choose the two mutually perpendicular lines as axes of coordinates.

The straight line segment AB of constant length 'p' slides so that A and B move along OX and OY respectively. Let $P(x_1, y_1)$ be a point of AB such that

$AP : PB = 1 : 2$. It is required to find the locus of P.



At any position of *AB*, let the intercepts *OA*, *OB* be *a*, *b* respectively, so that

$$a^2 + b^2 = p^2 \qquad \text{...(i)}$$

Since *AP* : *PB* = 1 : 2, we have

$$x_1 = \frac{2a}{3}, y_1 = \frac{b}{3}$$

$$\therefore \text{ a} = \frac{3x_1}{2}, b = 3y_1$$

Using (i) $\left(\frac{3x_1}{2}\right)^2 + (3y_1)^2 = p^2$, and this implies that

$P(x_1, y_1)$ should lie on the curve $\left(\frac{3x}{2}\right)^2 + (3y)^2 = p^2$

i.e., the required locus is $\frac{x^2}{4} + \frac{y^2}{1} = \frac{p^2}{9}$.

5 SHIFTING OF AXES

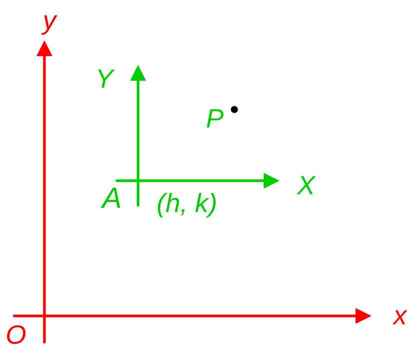
5.1 TRANSLATION OF COORDINATE AXES

When origin is shifted to a new position *A(h, k)* without changing the directions of the axes, then coordinates of all points in the plane are obtained by the formulae:

$x = X + h \quad y = Y + k$

where, *P* (*x, y*) referred to *ox, oy*

and *P*(*X, Y*) referred to *AX, AY*



5.2 ROTATION OF COORDINATE AXES

When coordinate axes are rotated in anti-clockwise direction through an angle θ, about the origin, then coordinates of a point *P* in the plane are obtained by the formulae.

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= X\sin\theta + Y\cos\theta \end{aligned}\right\} \text{ and } \left\{\begin{aligned} X &= x\cos\theta + y\sin\theta \\ Y &= y\cos\theta - x\sin\theta \end{aligned}\right.$$

Where *P* is *(x, y)* referred to coordinate system *oxy* and *P* is *(X, Y)* referred to new coordinate system *oXY.*

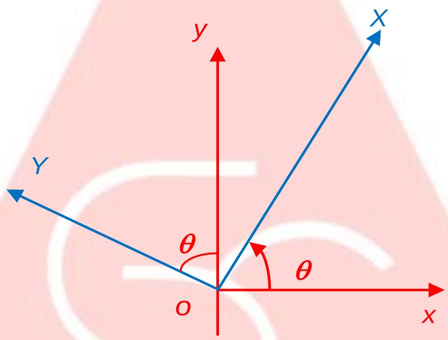


Illustration 6

Question: **Find the equation of the curve $x^2 + 2y^2 + 3x - 4y + 7 = 0$, when the origin is transferred to the point (1, –2) without changing the direction of axes.**

Solution: If *P(x, y)* be any point on the curve and (x_1, y_1) be the coordinates of *P. w. r.* to new axes then $x_1 = x - \alpha = x - 1$ and $y_1 = y - \beta = y + 2$

$\therefore x = x_1 + 1,\ y = y_1 - 2$

Hence new equation will be

$$(x_1 + 1)^2 + 2(y_1 - 2)^2 + 3(x_1 + 1) - 4(y_1 - 2) + 7 = 0$$

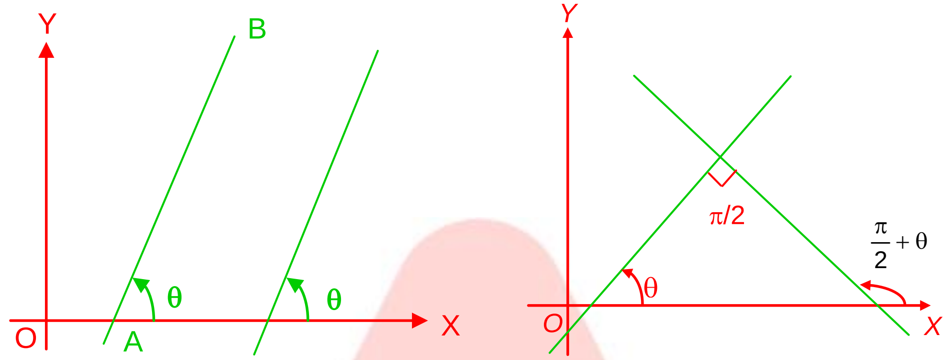
or $$x_1^2 + 2x_1 + 1 + 2(y_1^2 - 4y_1 + 4) + 3x_1 + 3 - 4y_1 + 8 + 7 = 0$$

or $$x_1^2 + 2y_1^2 + 5x_1 - 12y_1 + 27 = 0$$

Thus new equation of the curve will be

$$x^2 + 2y^2 + 5x - 12y + 27 = 0$$

Note: New equation of the curve can be directly obtained by putting $(x + 1)$ in place of x and $(y - 2)$ in place of y in the given equation of the curve.



If a straight line AB makes an angle θ with the positive direction of the x-axis, tan θ is called *the slope or gradient of the straight line* and is usually denoted by the letter 'm'.

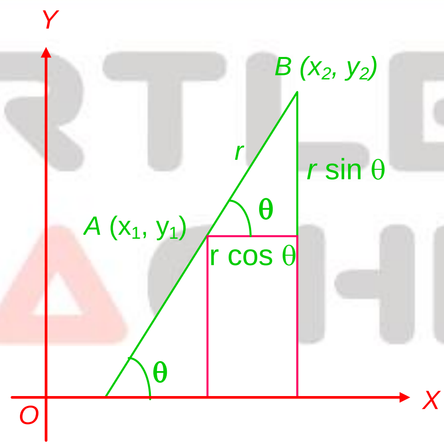
It follows that

(i) if two lines are parallel, their slopes are equal, for the lines must be equally inclined to the positive direction of the x-axis.

(ii) if two lines are perpendicular the product of their slopes is –1, for if one line is inclined at an angle θ to the x-axis, the other must be inclined at $\frac{\pi}{2}+\theta$, hence their slopes are tan θ and $\tan\left(\frac{\pi}{2}+\theta\right)$, i.e., tan θ and –cot θ.

∴ the product is –1.

6.1 TO FIND THE SLOPE OF THE LINE JOINING ANY TWO POINTS (x_1, y_1) AND (x_2, y_2)



Let the segment joining of the points be of length r and let the line be inclined to the x-axis at angle θ.

$$\therefore \quad r\cos\theta = x_2 - x_1$$

$$r\sin\theta = y_2 - y_1$$

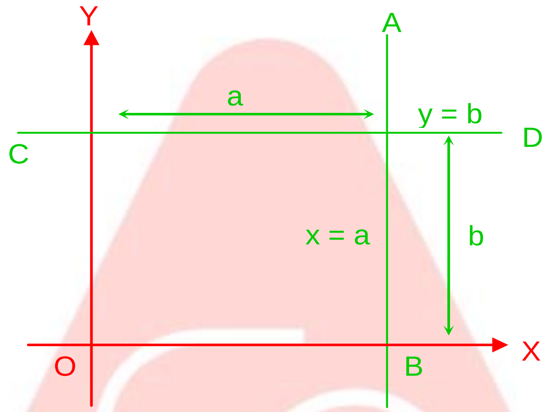
and therefore $\tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$

and tan θ is the required slope. The expression for the slope is, therefore,

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\text{difference of the ordinates of the two points}}{\text{difference of the abscissa}}$$

6.2 LINES PARALLEL TO THE CO ORDINATE AXES

Let the line *AB* be parallel to the *Y*-axis and at a distance '*a*' from it. Every point on *AB* will have its abscissa 'a', and hence the equation of *AB* is $x = a$. By putting $a = 0$, we deduce that the equation of the *y*-axis is $x = 0$.



Similarly, the equation of the straight line *CD* parallel to the *X*-axis and at a distance '*b*' from it is $y = b$. By putting $b = 0$ we deduce that the equation to the *X*-axis is $y = 0$

6.3 ANGLE BETWEEN TWO GIVEN STRAIGHT LINES

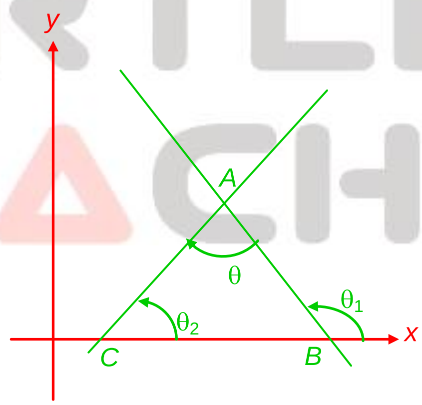
Let *AB, AC* have slopes m_1, m_2 and be inclined to the *X*-axis at θ_1, θ_2.

Then $\angle = \theta_1 - \theta_2 = \theta$

$$\therefore \tan \angle CAB = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

i.e., $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

where θ is the angle between *AB* and *AC*



Note: It is customary to take θ, as the acute angle between the two lines, and hence mostly one can take the above formula as $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

If the lines are parallel,

$\tan\theta = 0$, since $\theta = 0$

$\therefore m_1 - m_2 = 0$

$\therefore \boldsymbol{m_1 = m_2}$

and **if the lines are perpendicular,** tan θ is not defined, since $\theta = \frac{\pi}{2}$, and

therefore $m_1m_2 + 1 = 0$

$\therefore \boldsymbol{m_1m_2 = -1}$.

Illustration 7

Question: **Find the acute angle between the two lines with slopes $\frac{1}{5}$ and $\frac{3}{2}$.**

Solution: If the angle between the lines is θ,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\frac{1}{5} - \frac{3}{2}}{1 + \left(\frac{1}{5}\right)x\left(\frac{3}{2}\right)}\right| = |-1| = 1$$

Therefore the angle is 45°.

7 VARIOUS FORMS OF THE EQUATION OF A STRIGHT LINE

It was said earlier that every curve, in coordinate geometry, is represented by an equation which is a relation connecting *x* and *y,* the coordinates of any point which lies on that curve. In particular, any straight line has also an equation to represent it. It may be seen that to represent a straight line we need to be given two independent information; and depending upon the two, the equation also changes.

We have therefore the following forms of equations to straight lines:

7.1 SLOPE ONE POINT FORM

Given that *a* line has a slope (*m = tan θ)* which gives the direction it may be noted that '*m*' alone does not give the equation of the line and with the same slope there can be any number of straight lines all of which are parallel. Given that it passes through a given point (x_1, y_1). In this case the equation has the form.

$$\boldsymbol{y - y_1 = m(x - x_1)}$$

Illustration 8

Question: **If a line has a slope = $\frac{1}{2}$ and passes through *(– 1, 2);* find its equation.**

Solution: The equation of the line is

$$y - 2 = \frac{1}{2}(x - (-1))$$

i.e., $2y - 4 = x + 1$

i.e., $x - 2y + 5 = 0$

7.2 Y-INTERCEPT FORM

Given the slope '*m*' and the length '*c*' (called the intercept) cut off on the *y*-axis by the line. In this case the form of the equation is

$$\boldsymbol{y = mx + c}$$

Illustration 9

Question: **If a line has a slope $\frac{1}{2}$ and cuts off along the positive *y*-axis of length $\frac{5}{2}$ find the equation of the line.**

Solution: $y = \frac{1}{2}x + \frac{5}{2}$

i.e., $2y = x + 5$

i.e., $x - 2y + 5 = 0$

7.3 TWO POINT FORM

Equation of a line passing through two points (x_1, y_1) and (x_2, y_2).

In this case the equation to the line is of the form

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Where $x_1 \neq x_2$ and $\left(\frac{y_1 - y_2}{x_1 - x_2}\right)$ is the slope (m) of the line.

Illustration 10

Question: **If a line passes through two points (1, 5) and (3, 7), find its equation.**

Solution: $\frac{y-5}{x-1} = \frac{-7+5}{-3+1} = \frac{-2}{-2} = 1$

i.e., $y - 5 = x - 1$

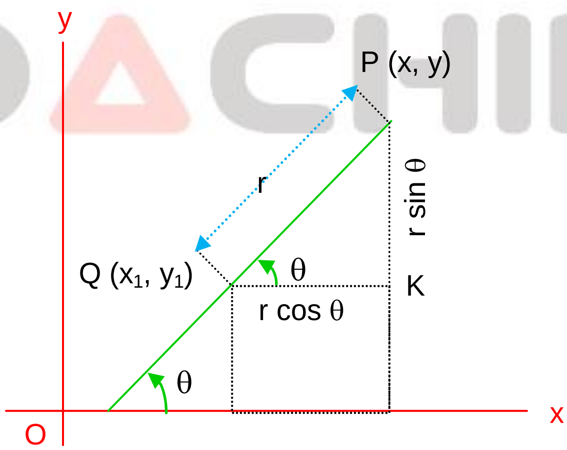
i.e., $x - y + 4 = 0$

7.3 SYMMETRIC FORM

If (x_1, y_1) is a given point on a straight line, (x, y) any point on the line, θ the inclination of the line and r the distance between the points (x, y) and (x_1, y_1) then $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}$

Let $P(x, y)$ represents any point on the line. $Q(x_1, y_1)$ is a given point, such that $PQ = r$.
Draw lines through Q and P parallel to QY, and draw QK parallel to OX.
From the triangle QKP.

$QK = PQ \, COS\, \theta$



or $x - x_1 = r\cos\theta$ and $\frac{x - x_1}{\cos\theta} = r$

Also $PK = PQ \sin\theta$ or $y - y_1 = r\sin\theta$ and $\frac{y - y_1}{\sin\theta} = r$

$$\therefore \ \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

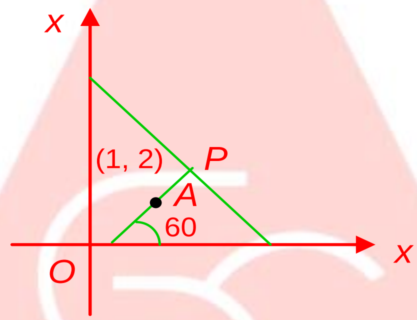
This is called the *symmetric form of the equation to a line* (This will be found very useful in cases where we have to deal with the length (*r*) of any portion of a line.)

Illustration 11

Question: **A straight line passes through a point A (1, 2) and makes an angle 60° with the x-axis. This line intersects the line $x + y = 6$ at the point *P*. Find AP.**

Solution: Let required line be

$$\frac{x - 1}{\cos 60^\circ} = \frac{y - 2}{\sin 6^\circ} = r (\equiv AP)$$



$\therefore$ *P* is $(1 + r\cos 60^\circ, 2 + r\sin 60^\circ)$ which lies on $x + y = 6$

The condition is

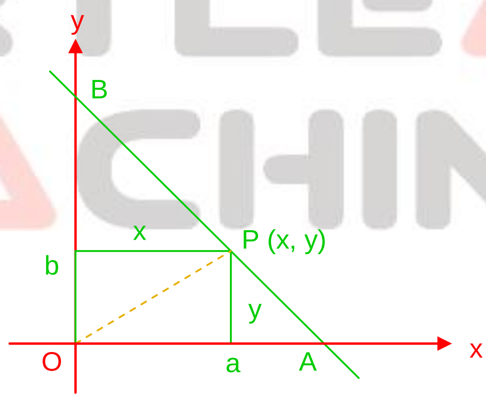
$$\left(1 + \frac{r}{2}\right) + \left(2 + r\frac{\sqrt{3}}{2}\right) = 6 \text{ or } r\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = 3$$

$$\therefore r = \frac{6}{1+\sqrt{3}} = \frac{6(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})} = \frac{6(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})} = \frac{6(1-\sqrt{3})}{-2} = 3(\sqrt{3}-1).$$

7.5 INTERCEPT FORM

Let the line make intercepts *a* and *b* on the axes of *x* and *y* respectively.

Let *P*(*x, y*) represent any point on the line. Join *OP.*



We have,

Area of triangle *OAB* = area of triangle *OAP* + area of triangle *OPB*

i.e., $\frac{1}{2}ab = \frac{1}{2}ay + \frac{1}{2}bx$

or $\frac{x}{a} + \frac{y}{b} = 1$

This is called the intercept form of the equation of straight line.

Illustration 12

Question: **Find the equation of the straight line, which passes through the point (3, 4) and whose intercept on *y*-axis is twice that on *x*-axis.**

Solution: Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \text{...(i)}$$

According to the question $b = 2a$

$\therefore$ from (i) equation of line will become

$$\frac{x}{a}+\frac{y}{2a}=1 \quad \text{or } 2x + y = 2a \qquad \text{....(ii)}$$

Since line (ii) passes through the point (3, 4)

$\therefore$ 2 x 3 + 4 = 2a $\qquad$ $\therefore$ $a = 5$

$\therefore$ from (ii), equation of required line will be

$2x + y = 10$.

7.6 NORMAL FORM

If p is the length of the perpendicular from the origin upon a straight line, and that α is the angle the perpendicular makes with the axis of x, equation of the straight line can be obtained as follows:

Let PQ be the straight line, OQ (= p) the perpendicular drawn to it from the origin O, and $\angle QOX = \alpha$

Draw the ordinate PN also draw NR perpendicular to OQ, and PM perpendicular to RN

We have $\angle PNM = 90^\circ - \angle RNO = \alpha$

$$\therefore \; p = OQ = OR + RQ = OR + PM$$

$$= ON \; \cos \; \alpha + PN \; \sin \; \alpha$$

or $p = x\cos\alpha + y\sin\;\alpha$

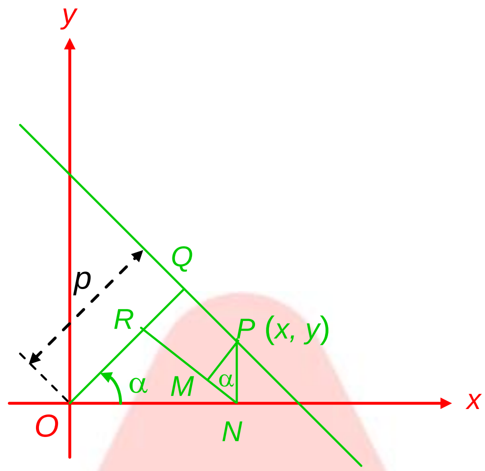
$\therefore$ $x \cos \alpha + y \sin \alpha = p$ is the required equation.

This is called the *perpendicular form*.

Alternatively,

Suppose $\frac{x}{a}+\frac{y}{b}=1$ is the equation of the straight line

In this case, $a=\frac{p}{\cos\alpha}, b=\frac{p}{\sin\alpha}$



∴ by substitution, the equation becomes

$$\frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1 \text{ or } \boldsymbol{x\cos\alpha + y\sin\alpha = p}$$

This is known as the normal form of the equation to a straight line.

Illustration 13

Question: **Find the equation of the straight line upon which the length of perpendicular from origin is $3\sqrt{2}$ units and this perpendicular makes an angle of 75° with the positive direction of *x*-axis.**

Solution: Let *AB* be the required line and *OL* be perpendicular to it.

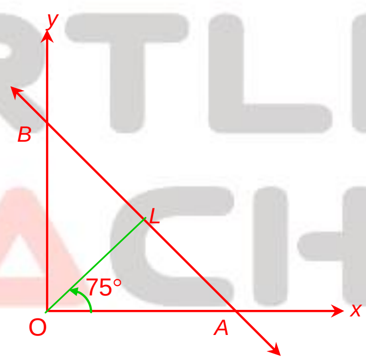
Given $OL = 3\sqrt{2}$ and $\angle LOA = 75°$

∴ equation of line *AB* will be

$x\cos 75° + y\sin 75° = 3\sqrt{2}$

[Normal form] ...(i)

Now $\cos 75° = \cos(30° + 45°) = \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} - \frac{1}{2}.\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$



and $\sin 75° = \sin(30° + 45°) = \sin 30°\cos 45° + \cos 30°\sin 45° = \frac{1}{2}.\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

∴ from (i) equation of line *AB* is

$$x\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + y\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = 3\sqrt{2} \text{ or } (\sqrt{3}-1)x + (\sqrt{3}+1)y - 12 = 0$$

7.7 GENERAL FORM

Any linear (1st degree in *x* and *y*) equation of the form

$Ax + By + C = 0$

represents a straight line.

The straight line in this form, has

(a) slope = $m = -\dfrac{A}{B} = -\dfrac{\text{coefficient of } x}{\text{coefficient of } y}$

(b) x intercept = $-\dfrac{C}{A}$ and

y intercept = $-\dfrac{C}{B}$

Illustration 14

Question: **Find the value of k so that the straight line $2x + 3y + 4 + k(6x - y + 12) = 0$ and $7x + 5y - 4 = 0$ are perpendicular to each other.**

Solution: Given lines are

$(2 + 6k)\, x + (3 - k)\, y + 4 + 12k = 0$ …(i)

and $7x + 5y - 4 = 0$ …(ii)

slope of line (i), $m_1 = -\dfrac{2+6k}{3-k} = \dfrac{2+6k}{k-3}$

and slope of line (ii), $m_2 = -\dfrac{7}{5}$

Since line (i) is perpendicular to line (ii)

$\therefore \left(\dfrac{2+6k}{k-3}\right)\left(-\dfrac{7}{5}\right) = -1$

or $(2 + 6k)\, 7 = 5\,(k - 3)$

or $14 + 42k = 5k - 15$

or $37k = -29$

or $k = -\dfrac{29}{37}$

8 POINT OF INTERSECTION OF TWO LINES

Consider two non-parallel lines : $ax + by + c = 0$ and $a'x + b'y + c' = 0$

$$i.e. -\frac{a}{b} \neq -\frac{a'}{b'} \Rightarrow ab' - a'b \neq 0$$

Their point of intersection can be obtained by solving their equations simultaneously which can be done through substitution, elimination or cross-multiplication rule.

The point of intersection is given by:

$$x = \frac{bc' - b'c}{ab' - a'b}, \quad y = \frac{a'c - ac'}{ab' - a'b}$$

Illustration 15

Question: **Show that the lines $2x - y - 12 = 0$ and $3x + y - 8 = 0$ intersect at a point which is equidistant from both the coordinate areas.**

Solution: $2x - y - 12 = 0$

$3x + y - 8 = 0$

Using cross multiplication rule, we have

$$\frac{x}{(-1)(-8)-(1)(-12)} = \frac{-y}{(2)(-8)-(3)(-12)} = \frac{1}{(2)(1)-(3)(-1)} \Rightarrow \frac{x}{20} = \frac{-y}{20} = \frac{1}{5}$$

$\Rightarrow x = \frac{20}{5} = 4$ and $y = -\frac{20}{5} = -4$

So, the point is (4, –4) which is at a distance 4 units from both the coordinate axes.

9 AREA OF TRIANGLE WHEN EQUATION OF SIDES ARE GIVEN

Let equations of sides be

$a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

$a_3x + b_3y + c_3 = 0$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\therefore$ area of triangle $ABC = \frac{\Delta^2}{2|C_1C_2C_3|}$

where C_1, C_2, C_3 are cofactors of c_1, c_2, c_3 in the determinant Δ.

Illustration 16

Question: **Find the area of triangle formed by the lines $x - y + 1 = 0$, $2x + y + 4 = 0$ and $x + 3 = 0$.**

Solution: Area $= \frac{1}{2|C_1C_2C_3|}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{vmatrix}^2$

where $C_1 = (2)(0) - (1)(1) = -1$

$C_2 = -[(1)(0) - (1)(-1)] = -1$

$C_3 = (1)(1) - (2)(-1) = 3$

Putting these values, we get, Area $= \frac{1}{2(3)}[1(-4-1) + 3(1+2)]^2$ (expanding along last row)

$= \frac{8}{3}$ sq. units

10 CONDITION OF COLLINEARITY

Three points *A, B* and *C* or more are said to be collinear when they lie on the same line. Collinearity can be checked in two simple ways.

(i) If Slope of *AB* = slope of *BC*, points *A, B, C* are collinear.

(ii) If Area of $\Delta ABC = 0$, points *A, B, C* are collinear.

(iii) Collinear points must satisfy section formulae.

Illustration 17

Question: **Prove that the points (1, 2), (-3, λ) and (4, 0) are collinear if λ is equal to $\frac{14}{3}$.**

Solution: Let *A* (1,2), *B* (-3, λ) and *C* (4, 0) to be the three points.

Slope of *AB* = slope of *AC*

$$\Rightarrow \frac{\lambda-2}{-3-1}=\frac{2-0}{1-4} \Rightarrow \frac{\lambda-2}{-4}=-\frac{2}{3}$$

$$\Rightarrow \lambda=2+\frac{8}{3}=\frac{14}{3}$$

Alternatively,

$$\text{Area } (\Delta ABC)=\frac{1}{2}\begin{vmatrix}1 & 2 & 1\\ -3 & \lambda & 1\\ 4 & 0 & 1\end{vmatrix}=0$$

$$\Rightarrow 4(2-\lambda)+(\lambda+6)=0$$

$$\Rightarrow -4\lambda+\lambda+8+6=0$$

$$\Rightarrow \lambda=\frac{14}{3}$$

11 CONDITION OF CONCURRENCY

Three or more lines are said to be concurrent if they all pass through the same point.

If the lines

$a_1 x + b_1 y + c_1 = 0$

$a_2 x + b_2 y + c_2 = 0$

$a_3 x + b_3 y + c_3 = 0$

are concurrent, the area of triangle formed by them is zero.

$$\therefore \begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=0$$

Illustration 18

Question: **The line $x + \lambda y - 4 = 0$ passes through the point of intersection of $4x - y + 1 = 0$ and $x + y + 1 = 0$. Find the values of λ.**

Solution: The three lines are concurrent

$$\Rightarrow \begin{vmatrix}1 & \lambda & -4\\ 4 & -1 & 1\\ 1 & 1 & 1\end{vmatrix}=0$$

$$\Rightarrow -2-3\lambda-20=0$$

$$\Rightarrow \lambda=-\frac{22}{3}$$

12 FAMILY OF LINES

Family means a group of lines having a particular characteristic. e.g., family of lines parallel to x-axis are of the form $y = k$, where k is any arbitrary real number. Family of lines perpendicular to $2x - y + 4 = 0$ is $x + 2y + k = 0$ since product of slopes has to be –1. so, we can say

(i) Family of lines parallel to $ax + by + c = 0$ is of the form $ax + by + k = 0$

(ii) Family of lines perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$

(iii) Family of lines through the point of intersection of the lines

$ax + by + c = 0$ ..(i)

and $a'x + b'y + c' = 0$ ...(ii)

can be written as

$(ax + by + c) + k\,(a'x + b'y + c') = 0$ ...(iii)

Equation (i) represents a straight line, for it is of the first degree.

Let (x_1, y_1) be the coordinates of the common point of (i) and (ii)

The point $(x_1, ,y_1)$ lies on (i);

$\therefore\ ax_1 + by_1 + c = 0$ ...(iv)

The point $(x_1, ,y_1)$ lies on (ii)

$\therefore\ a'x_1 + b'y_1 + c' = 0$ …(v)

Multiplying (v) by k, and adding to (iv), we get

$ax_1 + by_1 + c + k\left(a'x_1 + b'y_1 + c'\right) = 0$ ...(vi)

From this we see that the values x_1, y_1 satisfy the equation (iii)

$\therefore$ the straight line represented by (iii) passes through $(x_1, ,y_1)$ the common point of (i) and (ii).

Illustration 19

Question: **Find the equation of a line parallel to $x + 2y = 3$ and passing through the point (3, 4).**

Solution: Any line parallel to $x + 2y = 3$ is of the form $x + 2y = k$.
The point (3,4) satisfies this equation
$\Rightarrow 3 + 2(4) = k \Rightarrow k = 11$
So, required line is $x + 2y = 11$

Illustration 20

Question: **Find the equation of line perpendicular to $2x - 3y = 5$ and cutting off an intercept 1 on the x-axis.**

Solution: Any line perpendicular to $2x - 3y = 5$ is of the form $3x + 2y = k$
Putting $y = 0$, we get $x = \frac{k}{3}$, the x-intercept.
Since, $\frac{k}{3} = 1 \Rightarrow k = 3$
The required line is $3x + 2y - 3 = 0$

Illustration 21

Question: **Find the equation of the straight line passing through (2, –9) and the point of intersection of lines $2x + 5y - 8 = 0$, $3x - 4y - 35 = 0$.**

Solution: Any line through the point of intersection of given lines is $2x + 5y - 8 + K\,(3x - 4y - 35) = 0$
Point (2, –9) lies on it, implies $K = 7$.
$\therefore$ The required line becomes $x - y - 11 = 0$

Illustration 22

Question: **Find the equation of straight line passing through the point of intersection of lines $3x - 4y + 1 = 0$, $5x + y - 1 = 0$ and cutting off equal intercepts from coordinate axes.**

Solution: Any line passing through the point of intersection of given lines is

$$3x - 4y + 1 + K(5x + y - 1) = 0$$

or $(3 + 5K)x + (K - 4)y = K - 1$

Writing it in intercept form as

$$\frac{x}{\frac{K-1}{3+5K}} + \frac{y}{\frac{K-1}{K-4}} = 1, \text{ if } K \neq 1$$

For equal intercepts: $\frac{K-1}{K-4} = \frac{K-1}{3+5K}$ which gives $K = \frac{-7}{4}$

Hence the required equation is $23x + 23y = 11$

13 DISTANCE OF A POINT FROM A LINE

The perpendicular distance of a point from a line can be obtained when the equation of the line and the coordinates of the point are given.

Case I :

Let us first derive the formula for this purpose when the equation of the line is given in normal form.

Let the equation of the line l in normal form be

$x \cos \alpha + y \sin \alpha = p$,

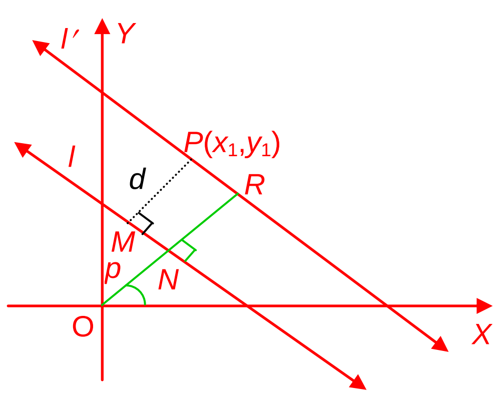
when α is the angle made by the perpendicular from origin to the line with positive direction of x-axis and p is the length of this perpendicular. Let $P(x_1, y_1)$ be the point, not on the line l. Let the perpendicular drawn from the point P to the line l be PM and $PM = d$. Point P is assumed to lie on opposite side of the line l from the origin O. Draw a line l' parallel to the line l through the point P. Let ON be perpendicular from the origin to the line l which meets the line l' in point R.

Obviously

$ON = p$ and $\angle XON = \alpha$

Also, from the Figure, we note that

$OR = ON + NR = p + MP$



Therefore, length of the perpendicular from the origin to the line l' is

$OR = p + d$

and the angle made by the perpendicular OR with positive direction of x-axis is α. Hence, the equation of the line l' in normal form is

$x \cos \alpha + y \sin \alpha = p + d$

Since the line l' passes through the point P, the coordinates (x_1, y_1) of the point P should satisfy the equation of the line l' giving

$x_1 \cos \alpha + y_1 \sin \alpha = p + d$,

or $d = x_1 \cos \alpha + y_1 \sin \alpha - p$.

The length of a segment is always non–negative. Therefore, we take the absolute value of the RHS, i.e.,

$d = | x_1 \cos \alpha + y_1 \sin \alpha - p |$.

Thus, the length of the perpendicular is the absolute value of the result obtained by substituting the coordinates of point P in the expression $x \cos \alpha + y \sin \alpha - p$

Case II:

Let the equation of the line be

$Ax + By + C = 0$

Reducing the general equation to the normal form, we have

$$\pm \frac{A}{\sqrt{A^2+B^2}} x \pm \frac{B}{\sqrt{A^2+B^2}} y = \pm \frac{C}{\sqrt{A^2+B^2}}$$

Where sign is taken + or – so that the RHS is positive.

(a) When $C < 0$

In this case the normal form of the equation of the line / becomes

$$\frac{A}{\sqrt{A^2+B^2}} x + \frac{B}{\sqrt{A^2+B^2}} y = -\frac{C}{\sqrt{A^2+B^2}}$$

Now, from the result of case (I), the length of the perpendicular segment drawn from the point $P(x_1, y_1)$ to the line (II) is

$$d = \left| \frac{A}{\sqrt{A^2+B^2}} x_1 + \frac{B}{\sqrt{A^2+B^2}} y_1 + \frac{C}{\sqrt{A^2+B^2}} \right|$$

i.e., $$d = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}} \right|$$

(b) When $C > 0$

In this case the normal form of the equation (II) becomes

$$-\frac{A}{\sqrt{A^2+B^2}}x-\frac{B}{\sqrt{A^2+B^2}}y=\frac{C}{\sqrt{A^2+B^2}}$$

Again, by the result of case (I), the distance *d* is

$$d=\left|\frac{-Ax_1-By_1-C}{\sqrt{A^2+B^2}}\right|$$

or $$d=\left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right|$$

The perpendicular distance of (x_1, y_1) from the line $ax + by + c = 0$ is given by

$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

So, Perpendicular distance from origin is $\frac{|c|}{\sqrt{a^2+b^2}}$.

Illustration 23

Question: **The equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is $(2, -1)$. Find the length of side of the triangle.**

Solution: Equation of side *BC* is

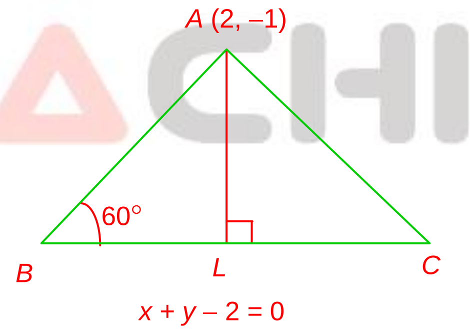
$x + y - 2 = 0$ …(i)

$A \equiv (2, -1)$

Now $AL = \left|\frac{2-1-2}{\sqrt{1^2+1^2}}\right| = \frac{1}{\sqrt{2}}$

From ΔABL, $\sin 60° = \frac{AL}{AB}$

$\therefore \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}AB}$ or $AB = \sqrt{\frac{2}{3}}$



14 DISTANCE BETWEEN PARALLEL LINES

The distance between $ax + by + c = 0$ and $ax + by + c' = 0$

is given by, $d = \frac{|c-c'|}{\sqrt{a^2+b^2}}$

Make sure that coefficients of x and y are same in both the lines before applying the formula.

Illustration 24

Question: **Find the distance between the lines $5x + 12y + 40 = 0$ and $10x + 24y - 25 = 0$.**

Solution: Here, coefficients of x and y are not the same in both the equations. So, we write them as

$5x + 12y + 40 = 0$

$5x + 12y - \frac{25}{2} = 0$

Now, distance between them = $\frac{\left|40 - \left(-\frac{25}{2}\right)\right|}{\sqrt{(5)^2 + (12)^2}} = \frac{105}{2(13)} = \frac{105}{26}$

15 IMAGE OF A POINT IN A LINE

The point Q situated at the same distance behind the line mirror $ax + by + c = 0$ as the point P in front of it, is called the image of P.

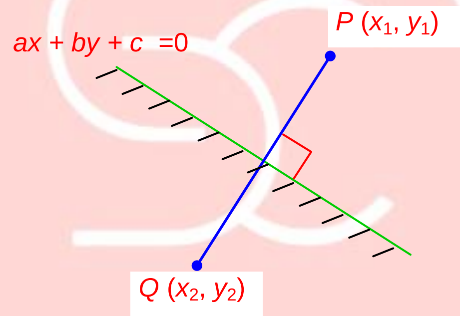


Illustration 25

Question: **If P is (1,2) and the line mirror is $2x - y + 4 = 0$, find the coordinates of its image (*i.e.*, Q).**

Solution: Let the coordinates of Q be (x_2, y_2).

The mid-point of PQ lies on $2x - y + 4 = 0$

$\Rightarrow 2\left(\frac{x_2 + 1}{2}\right) - \left(\frac{y_2 + 2}{2}\right) + 4 = 0$

$\Rightarrow 2x_2 + 2 - y_2 - 2 + 8 = 0$

$\Rightarrow 2x_2 - y_2 + 8 = 0$ ...(i)

Also the line PQ is perpendicular to the line mirror.

$\Rightarrow 2y_2 - 4 + x_2 - 1 = 0$

$\Rightarrow x_2 + 2y_2 - 5 = 0$ ....(ii)

Solving (i) and (ii), we get

$\frac{x_2}{5 - 16} = \frac{-y_2}{-10 - 8} = \frac{1}{4 + 1}$

$\Rightarrow x_2 = -\frac{11}{5},\ y_2 = \frac{18}{5}$

$\therefore$ *i.e.* $\left(-\frac{11}{5}, \frac{18}{5}\right)$

16 POSITION OF A POINT WITH RESPECT TO A LINE

Two points (x_1 , y_1) and (x_2 , y_2) are located on the same or opposite sides of the line $ax + by + c = 0$ according as the sign of $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ is same or opposite.

e.g., the points (2, -3) and (4, 2) are on the same side of the line $6x – y + 2 = 0$ as $6(2) – (–3) + 2 = 17$ and $6 (4) –(2) + 2 = 24$ are of same sign (positive here).

On these lines, we can say that a point (x_1 , y_1) lies on the origin side of $ax + by + c = 0$ (*i.e.*, the side in which (0, 0) lies) if $ax_1 + by_1 + c$ is of the same sign as c.

Illustration 26

Question: **Find the values of λ for which the point (2 –λ, 1 + 2λ) lies on the non-origin side of the line $4x – y – 2 = 0$.**

Solution: Since 'c' is negative, the expression obtained by putting the coordinates of the point In the equation of line must be positive for the point to lie on non-origin side of the line.

$\therefore \quad 4(2 - \lambda) – (1 + 2\lambda) – 2 > 0$

$\Rightarrow \quad 5 – 6\lambda > 0$

$\Rightarrow \quad \lambda < \frac{5}{6}$

Illustration 27

Question: **Determine all values of a for which the point *(a, a²)* lies inside the triangle formed by the lines $x – y + 1 = 0$; $x + y – 3 = 0$ and $y + 2x = 0$.**

Solution: *ABC* is the triangle formed by the three lines and $P(a, a^2)$ lies insi41-45
de the triangle *ABC.*

Refer to the figure on the next page

1. For $x + y – 3 = 0$; (*0, 0*) and (a, a^2) lie on the same side.

$$a^2 + a - 3 < 0$$

i.e., $\frac{-1-\sqrt{13}}{2} < a < \frac{-1+\sqrt{13}}{2}$ …(i)

2. For $x – y + 1 = 0$; (*0, 0*) and (a, a^2) lie on the opposite sides.

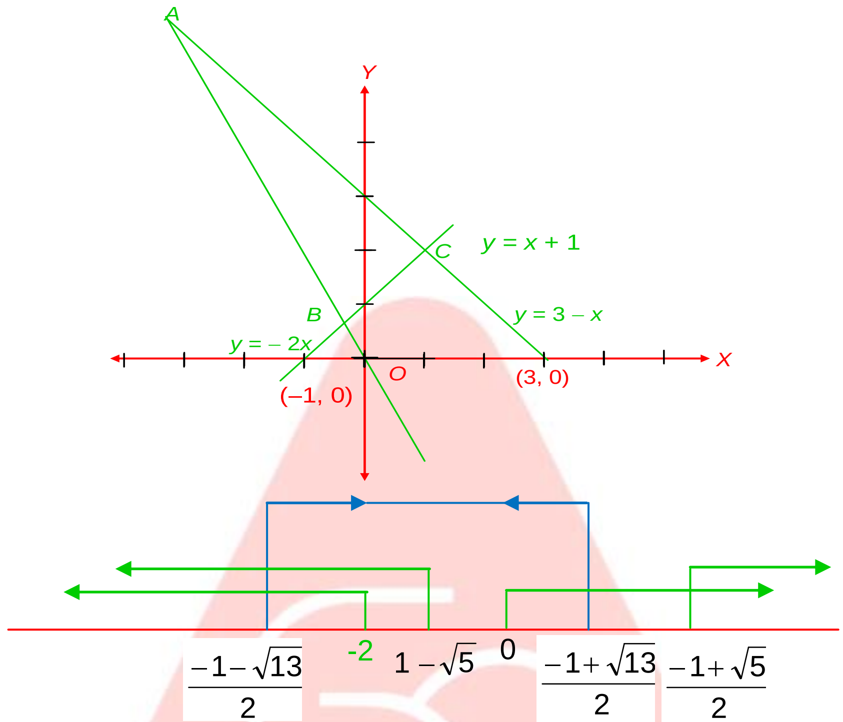
$$a - a^2 + 1 < 0$$

$\therefore \; a < \frac{1-\sqrt{5}}{2}$ (or) $a > \frac{1+\sqrt{5}}{2}$ ...(ii)

3. For the line $y + 2x = 0$, we have $2a + a^2 > 0$

i.e., $a > 0$ (or) $a < – 2$ ...(iii)

Combining the three conditions, the values of a are those satisfying $\frac{-1-\sqrt{13}}{2} < a < -2$



17 SOME POINTS AND THEIR COORDINATES RELATED TO A TRIANGLE

For a triangle having vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, the sides opposite to these vertices being *a, b* and *c* respectively, we state the following results:

17.1 CENTROID

Centroid, G : Point of intersection of medians

$$G \text{ is } \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

17.2 INCENTRE

Incentre, I : Point of intersection of internal angle bisectors

$$I \text{ is } \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

17.3 EXCENTRES

Excentres, I_1, I_2, I_3 Point of intersection of two exterior angle bisectors and the remaining internal angle bisector. Excentre opposite angle *A* is denoted by I_1 and its coordinates are

$$\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

17.4 CIRCUMCENTRE

Circumcentre, S : Point of intersection of perpendicular bisectors of sides

17.5 ORTHOCENTRE

Orthocentre, H : Point of intersection of altitudes

For an equilateral triangle all these points coincide.

For any other triangle G, H and S are collinear and G divides HS internally in the ratio 2 : 1.

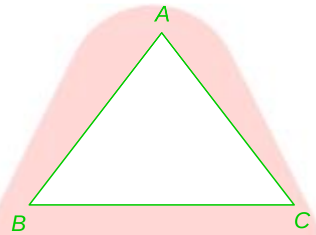
Illustration 28

Question: **Find the incentre *I* of ΔABC if *A* is (4, –2), *B* is (–2, 4) and *C* is (5, 5).**

Solution: $BC \equiv a = \sqrt{(5+2)^2 + (5-4)^2} = 5\sqrt{2}$

$CA \equiv b = \sqrt{(5-4)^2 + (5+2)^2} = 5\sqrt{2}$

$AB \equiv c = \sqrt{(-2-4)^2 + (4+2)^2} = 6\sqrt{2}$



If incentre *I* is $(\bar{x}, \bar{y})$, then, by the formula,

$$\bar{x} = \frac{ax_1 + bx_2 + cx_3}{a+b+c} = \frac{20\sqrt{2} - 10\sqrt{2} + 30\sqrt{2}}{5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}} = \frac{5}{2}$$

$$\bar{y} = \left(\frac{ay_1 + by_2 + cy_3}{a+b+c}\right) = \frac{-10\sqrt{2} + 20\sqrt{2} + 30\sqrt{2}}{5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}} = \frac{5}{2}$$

$\therefore$ Incentre is $\left(\frac{5}{2}, \frac{5}{2}\right)$.

Illustration 29

Question: **Find the coordinates of the orthocentre of the triangle whose vertices are (0, 0), (2, –1) and (–1, 3).**

Solution: Let $A \equiv (0, 0)$, $B \equiv (2, -1)$ and $C \equiv (-1, 3)$

$\left[\because \text{ Slope of } BC = \frac{-1-3}{2+1}\right]$

Since $AL \perp BC$, therefore equation of line *AL* is

$y - 0 = \frac{3}{4}(x - 0)$

or $4y = 3x$ or $3x - 4y = 0$ ...(i)

Slope of *AC* = –3

Since $BM \perp AC$, therefore, equation of line *BM* is

$y + 1 = \frac{1}{3}(x - 2)$ or $3(y + 1) = x - 2$

or $x - 3y = 5$ ….(ii)

Solving equations (i) and (ii), we get $x = -4$, $y = -3$

$\therefore$ Orthocentre (–4, –3).

18 ANGLE BISECTORS

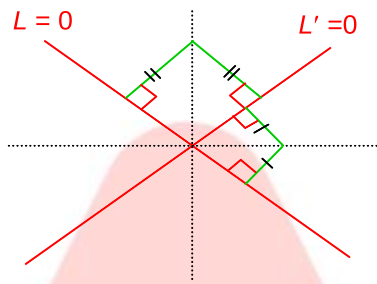
The locus of a point which moves such that its distance from two intersecting lines is same lies on the angle bisector of the two lines.

Let the two lines be

L : $ax + by + c = 0$

$L' : a'x + b'y + c' = 0$

$$\therefore \frac{|ax+by+c|}{\sqrt{a^2+b^2}} = \frac{|a'x+b'y+c'|}{\sqrt{a'^2+b'^2}}$$



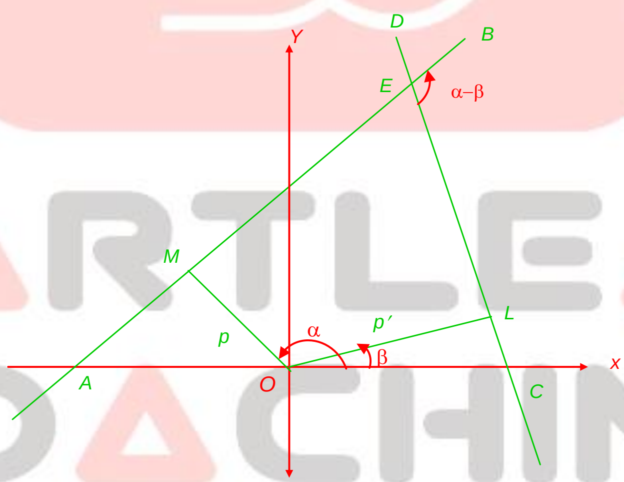
Clearly, there are two angle bisectors.

Methods to distinguish between acute angle bisector and obtuse angle bisector.

(i) Take any bisector and take any line. Find the acute angle between them. If it is less than 45°, the bisector taken lies in the acute angle region otherwise in the obtuse angle region.

(ii) The two lines $ax + by + c = 0$ and $a' x + b' y + c' = 0$ divide the xy plane (the coordinate axes plane) into acute angle region and obtuse angle region. It is proposed to ascertain that, in which of these two regions, the origin is situated.

Without loss of generality, the two equations may be so adjusted that both c and c' are positive.



Let AB and CD be the two given lines intersecting at E. p and p' are the lengths of the perpendiculars from O on AB and CD respectively.

$$\therefore p = \frac{c}{\sqrt{a^2+b^2}} \text{ and } p' = \frac{c'}{\sqrt{a'^2+b'^2}}$$

Let α and β be the angles made by p and p' with positive OX. Let $\alpha > \beta$.

Then $\alpha - \beta$ is the angle contained between p and p' and it is easy to see that $\angle CEB = \alpha - \beta$ (since $OLEM$ is cyclic) and this angle $\alpha - \beta$ is the supplement of the angle between the given lines in which the origin is situated. In other words, the origin is situated in that region of the angle between the two given lines, this angle being $180° - (\alpha - \beta)$.

Case I:

If $\alpha - \beta$ is acute, the origin is situated in the obtuse angle region and in that case

$\cos(\alpha - \beta)$ is positive.

Case II:

If $\alpha - \beta$ is obtuse, then the origin is situated in the acute angle region and in this case $\cos(\alpha - \beta)$ is negative.

The first equation $ax + by + c = 0$ may be written as

$$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y + \frac{c}{\sqrt{a^2+b^2}} = 0 \text{ and}$$

identifying this with $-x\cos\alpha - y\sin\alpha + p = 0$ $\left(\text{Note } p = \frac{c}{\sqrt{a^2+b^2}}\right)$

we have, $\cos\alpha = \frac{-a}{\sqrt{a^2+b^2}}$

$\sin\alpha = \frac{-b}{\sqrt{a^2+b^2}}$

Similarly we have, $\cos\beta = \frac{-a'}{\sqrt{a'^2+b'^2}}$

$\sin\beta = \frac{-b'}{\sqrt{a'^2+b'^2}}$

and $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$= \frac{aa'+bb'}{\sqrt{(a^2+b^2)(a'^2+b'^2)}}$$

Again, consider two cases,

$\therefore$ In the case (i) referred to above $\cos(\alpha - \beta) > 0$

$\therefore$ $aa' + bb' > 0$

and in this case origin is in the obtuse angle region.

In case (ii) $\cos(\alpha - \beta) < 0$

$\therefore$ $aa' + bb' < 0$

and in this case origin lies in the acute angle region. c is assumed to be positive and hence $ax + by + c > 0$ for the origin region. Similarly $a'x + b'y + c' > 0$ for the region in which the origin is contained.

The bisector of the angle in which the origin is situated is

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}} \quad \text{...(i)}$$

and this is the acute angle bisector if $aa' + bb' < 0$; and is the obtuse angle bisector if $aa' + bb' > 0$; and the bisector of the angle in which the origin is not situated is

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = -\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}} \quad \text{...(ii)}$$

and this is the acute angle bisector if $aa' + bb' > 0$; and is the obtuse angle bisector if $aa' + bb' < 0$.

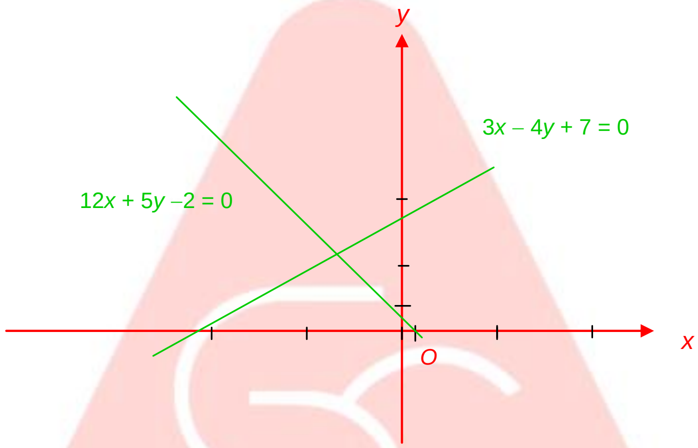
Illustration 30

Question: **Find the equation of the bisector of the acute angle between the lines $3x - 4y + 7 = 0$ $12x + 5y - 2 = 0$. Draw a rough sketch.**

Solution: The bisectors of the angle between the given lines are given by

$$\frac{3x-4y+7}{\sqrt{9+16}} = \pm\frac{12x+5y-2}{\sqrt{144+25}}$$

i.e., $13(3x - 4y + 7) = \pm 5(12x + 5y - 2)$



From the Figure, the bisector of the acute angle between the lines makes acute angle with the direction *OX* and hence its gradient is positive.

$\therefore$ the required bisector is given by

$13(3x - 4y + 7) = -5(12x + 5y - 2)$

i.e., $99x - 27y + 81 = 0$

i.e., $11x - 3y + 9 = 0$

Note: The other equation gives a line of negative gradient, and it represents the bisector of obtuse angle between the lines.

Alternatively, the equations of the lines may be written as

$3x - 4y + 7 = 0$

$-12x - 5y + 2 = 0$(Keeping the constant term positive)

Here, we have $a_1a_2 + b_1b_2 = -36 + 20 = -16$

i.e., $a_1a_2 + b_1b_2 < 0$, and hence the origin is situated in one of the acute angle regions.

$\therefore$the required bisector is given by $\frac{3x-4y+7}{5} = \frac{-12x-5y+2}{13}$

i.e., $13(3x - 4y + 7) = 5(-12x - 5y + 2)$

i.e., $99x - 27y + 81 = 0$ or $11x - 3y + 9 = 0$

Again, alternatively, the equations of the bisectors of the angles between the straight lines $3x - 4y + 7 = 0$, $12x + 5y - 2 = 0$ are

$$\frac{3x-4y+7}{5} = \pm\frac{12x+5y-2}{13}$$

i.e., $21x + 77y - 101 = 0$ and $11x - 3y + 9 = 0$ …(i) and (ii)

The following rule may be used to distinguish between these two lines.

Consider one of the given lines and one of the bisectors. We may take $3x - 4y + 7 = 0$ and $11x - 3y + 9 = 0$. Determine the tangent of the angle between these two lines.

The slope of the line $3x - 4y + 7 = 0$ is $\frac{3}{4}$

and the slope of the line $11x - 3y + 9 = 0$ is $\frac{11}{3}$

Let θ be the angle between these lines.

Then $\tan\theta = \frac{\frac{11}{3} - \frac{3}{4}}{1 + \frac{11}{3}.\frac{3}{4}} = \frac{35}{45} = \frac{7}{9}$ *i.e.,* $\tan\theta < 1$, and hence $\theta < 45°$

$\therefore$ we conclude that $11x - 3y + 9 = 0$ is the equation of the bisector of the acute angle between the given lines.

Note: The other bisector has therefore equation $21x + 77y - 101 = 0$

Illustration 31

Question: **The sides of a rhombus *ABCD* are parallel to the lines $x - y + 2 = 0$ and $7x - y - 3 = 0$. The diagonals intersect at (2, 3). The vertex *A* is on the *y-axis.* Find the possible positions of *A* on the y-axis.**

Solution: Let *A* be $(0, x_1)$

The sides of the rhombus are parallel to the lines $x - y + 2 = 0$ and $7x - y - 3 = 0$ and hence the diagonals of the rhombus are parallel to the bisectors of the angles between these lines. The bisectors have their slopes equal to the slopes of the diagonals.

The bisectors are $\frac{x - y + 2}{\sqrt{2}} = \pm\frac{7x - y - 3}{\sqrt{50}}$

i.e. $5x - 5y + 10 = 7x - y - 3$ *i.e.* $2x + 4y - 13 = 0$

or $5x - 5y + 10 = -7x + y + 3$ *i.e.* $12x - 6y + 7 = 0$

The slope of a diagonal joining $(0, x_1)$ and (2, 3) is $\frac{3 - x_1}{2 - 0}$ and this is $= -\frac{1}{2}$ or 2

$3 - x_1 = -1$ (or) $3 - x_1 = 4$

$x_1 = 4$ $\quad x_1 = -1$

The possible positions of *A* are (0, 4) or (0, –1)

19 HOMOGENEOUS EQUATION OF THE 2ND DEGREE

The equation $ax^2 + 2hxy + by^2 = 0$, homogeneous and of the second degree always represents a pair of straight lines through the origin. If the lines are taken as $y = mx$ and $y = m'x$ then it is identical to $(y - mx)(y - m'x) = 0$

Comparing,

$m + m' = \frac{-2h}{b}$ and $mm' = \frac{a}{b}$

Therefore the angle between the lines is given by

$\tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$, and

(i) when $h^2 = ab$, the two lines coincide and

(ii) when $a + b = 0$, the lines are perpendicular.

20 NON-HOMOGENEOUS 2ND DEGREE EQUATION

The general equation of 2nd degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of straight lines only if

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or equivalently in the determinant form $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

In that case, when representing two straight lines, the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

may be written as $(lx + my + n)(l'x + m'y + n') = 0$

so that the individual equations to the straight lines are

$lx + my + n = 0,\ l'x + m'y + n' = 0$

Moreover the lines are real if $h^2 \geq ab$.

The two equations compared: When $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, with $\Delta = 0$, represents a pair of straight lines; then $ax^2 + 2hxy + by^2 = 0$ represents two straight lines through the origin and respectively parallel to the other two lines.

It is, therefore, that when

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c \equiv (lx + my + n)(l'x + m'y + n') = 0$

represents two straight lines through a point say (x_1, y_1); then

$ax^2 + 2hxy + by^2 \equiv (lx + my)(l'x + m'y) = 0$

represents two straight lines through the origin respectively parallel to the first pair.

Note : The angle formula in Article 19 holds here also.

Illustration 32

Question: **Find straight lines represented by $6x^2 + 13xy + 6y^2 + 8x + 7y + 2 = 0$ and also find the point of intersection.**

Solution: $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\equiv 6(6)2 + 2\left(\frac{7}{2}\right)4\left(\frac{13}{2}\right) - 6\left(\frac{7}{2}\right)^2 - 6(4)^2 - 2\left(\frac{13}{2}\right)^2 = 254 - 254 = 0$$

identically. Hence given equation represents a pair of lines.

To find the lines, we rewrite the given equation as $6x^2 + (13y + 8)x + (6y^2 + 7y + 2) = 0$,

(a quadratic in x)

Solve for x:

$$x = \frac{-(13y + 8) \pm \sqrt{(13y + 8)^2 - 24(6y^2 + 7y + 2)}}{12}$$

$$= \frac{-(13y + 8) \pm (5y + 4)}{12}$$

$= \frac{-(2y+1)}{3}, \frac{-(3y+2)}{2}$

Hence lines are $3x + 2y + 1 = 0$ ....(i)

$2x + 3y + 2 = 0$ ....(ii)

Solve equation (i) & (ii) for the point of intersection, we get

$x = \frac{1}{5}, y = -\frac{4}{5}$

Thus lines intersect at point.

Illustration 33

Question: **Show that $bx^2 - 2hxy + ay^2 = 0$ represent a pair of straight lines, which are at right angles to the pair of lines given by $ax^2 + 2hxy + by^2 = 0$**

Solution: Let $ax^2 + 2hxy + by^2 = 0$ represent the lines $y - m_1x = 0$ and $y - m_2x = 0$

$\therefore \quad ax^2 + 2hxy + by^2 \equiv (y - m_1x)(y - m_2x)$

$\Rightarrow \quad m_1m_2x^2 - (m_1 + m_2)xy + y^2 \equiv ax^2 + 2hxy + by^2$

Comparing the coefficients on both sides,

$$\frac{m_1m_2}{a} = \frac{m_1 + m_2}{-2h} = \frac{1}{b}$$

or $\quad m_1 + m_2 = -\frac{2h}{b}, m_1m_2 = \frac{a}{b}$ ...(i)

(In other words, m_1, m_2 are roots of the quadratic equation $bm^2 + 2hm + a = 0$.

Since required lines are perpendicular to the given lines.

$$y = -\frac{1}{m_1}x \text{ and } y = -\frac{1}{m_2}x$$

Required pair of lines is $\left(y + \frac{x}{m}\right)\left(y + \frac{1}{m_2}x\right) = 0$ or $(m_1y + x)(m_2y + x) = 0$

$$m_1m_2y^2 + (m_1 + m_2)xy + x^2 = 0$$

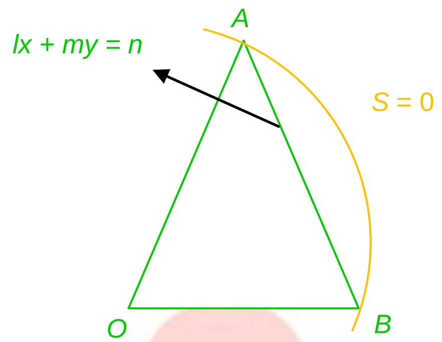
Using (i); $\frac{a}{b}y^2 - \frac{2h}{b}xy + x^2 = 0$ or $ay^2 - 2hxy + bx^2 = 0$

21 HOMOGENISATION

Let $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be a non-homogeneous second-degree equation. It generally represents a curve. [It may be a pair of straight lines also.]

The equation of a straight line be

$lx + my = n$ [first degree equation].



Now $S = 0$ and the above line generally meet at two points, say *A* and *B.*
Let *O* be the origin.
We get the combined equation of *OA* and *OB* by homogenizing

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

with the help of $\frac{lx + my}{n} = 1.$

This is because the equation to a pair of straight lines through the origin, is a homogeneous equation.

Homogenisation is done as

$$ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx + my}{n}\right) + 2fy\left(\frac{lx + my}{n}\right) + c\left(\frac{lx + my}{n}\right)^2 = 0.$$

Illustration 34

Question: **Prove that the straight lines joining the origin to the points of intersection of $x - y = 2$ and the curve $5x^2 + 12xy - 8y^2 + 8x - 4y + 12 = 0$ make equal angles with the axes.**

Solution: Using $x - y = 2$, homogenise the second degree equation to give

$$(5x^2 + 12xy - 8y^2)(2)(2) + (8x - 4y)(2)(x - y) + 12(x - y)(x - y) = 0$$

This simplified gives

$$x^2(12 + 16 + 12) + xy(48 - 16 - 8 - 24) + y^2(-32 + 8 + 12) = 0$$

$$48x^2 - 12y^2 = 0 \quad i.e., \ 4x^2 - y^2 = 0$$

This represents the two lines $y = 2x$ (slope $= 2$) and $y = -2x$ (slope = 2). These are equally inclined to the axes.

MIND MAP

Points and Straight Lines

- Distance Formula : $D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- Section Formula : $x = \frac{mx_2 + nx_1}{m+n}, y = \frac{my_2 + ny_1}{m+n}$
- Area of Triangle : $\Delta = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$
- Slope : $m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}, x_1 \neq x_2$
  $0 \le \theta < \pi$
- *A,B & C* are collinear if slope *AB* = Slope *AC* or Area (Δ *ABC*) = 0

Forms of Straight Lines:

- Slope one point form : $y - y_1 = m(x - x_1)$
- Two point form : $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$
- Intercept form : $\frac{x}{a} + \frac{y}{b} = 1$
- *y*-intercept form: $y = mx + c$
- Normal form : $x\cos\theta + y\sin\theta = p$
- Parametric form : $x = x_1 + r\cos\theta$
  $y = y_1 + r\sin\theta$
  $|r|$ = distance between (x, y) and (x_1, y_1)
- General form : $ax + by + c = 0$

Acute Angle between lines having slope m_1 and m_2:

$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

- Parallel if $m_1 = m_2$
- Perpendicular if $m_1 m_2 = -1$

- Area of triangle formed by $a_i x + b_i y + c_i = 0$, $i = 1, 2, 3$
  $\Delta = \frac{1}{|2C_1C_2C_3|}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$
- Three lines are concurrent if
  $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Family of Lines:

- Parallel to $ax + by + c = 0$ is $ax + by + k = 0$
- Perpendicular to $ax + by + c = 0$ is $bx - ay + k = 0$
- Passing though point of intersection of $L = 0$ & $L' = 0$ is $L + kL' = 0$

Joint equation of straight lines:

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

where $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,\ h^2 \ge ab$

Acute Angle between them:

$\theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$

- Distance of (x_1, y_1) from $ax + by + c = 0$
  $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$
- Distance between parallel lines
  $d = \frac{|c - c'|}{\sqrt{a^2 + b^2}}$

- **Angle Bisectors :**
  $\frac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$
  If $aa' + bb' < 0$: '+' sign gives acute angle bisector, if $aa' + bb' > 0$: '–' sign gives acute angle bisector, provided c & c' are positive.

- (x_1, y_1) & (x_2, y_2) are on same/opposite of $ax + by + c = 0$ if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of same/opposite sign.
- (x_1, y_1) & (x_2, y_2) are on origin/non-origin side of $ax + by + c = 0$ if $ax_1 + by_1 + c$ is of same/opposite sign to that of c.

Homogenisation :

Joint equation of lines joining origin to points of intersection of $lx + my + n = 0$ with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is given by

$ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$

Points related to a triangle:

- Centroid (G) : $\left(\frac{\Sigma x_1}{3}, \frac{\Sigma y_1}{3}\right)$
- Incentre (I) : $\left(\frac{\Sigma a x_1}{\Sigma a}, \frac{\Sigma a y_1}{\Sigma a}\right)$
- Excentre $I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c}\right)$ *etc.*
- Orthocentre (H) : altitudes
- Circumcentre (S) : Perpendicular bisectors

G,H,S are collinear & *HG* : *GS* = 2 : 1 except in equilateral triangle.