THREE DIMENSIONAL GEOMETRY

1 INTRODUCTION

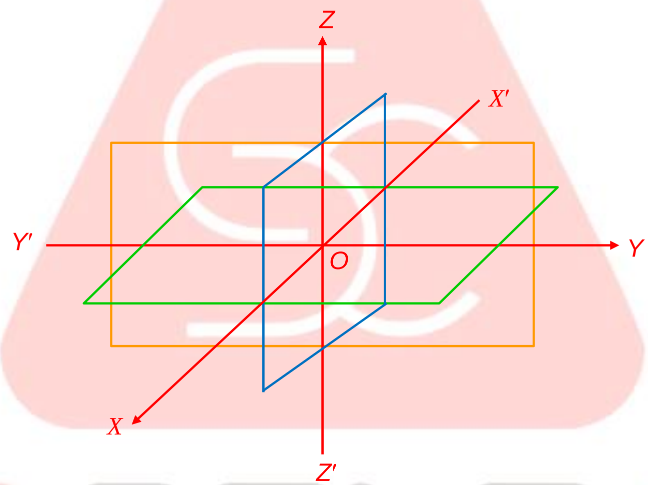
To locate a point in a plane we need a two dimensional coordinate system. On similar lines to locate a point in space we use a three dimensional coordinate system.

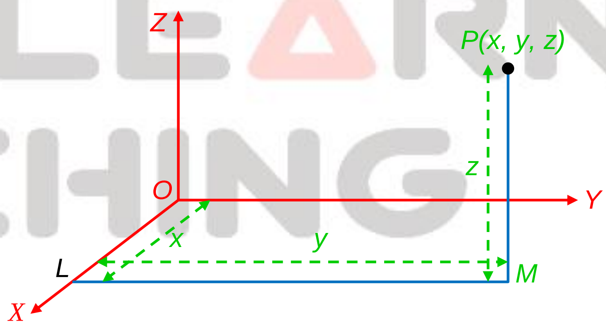
Consider three planes intersecting at a point *O*. The coordinate axes *x*-axis, y-axis and z-axis are shown by dark lines.

i.e., if we have an insect on the floor of a room, we can specify its position by a two dimensional coordinate system but if a spider is hanging from the roof of a room we specify its position in a three dimensional coordinate system.

2 COORDINATE AXES

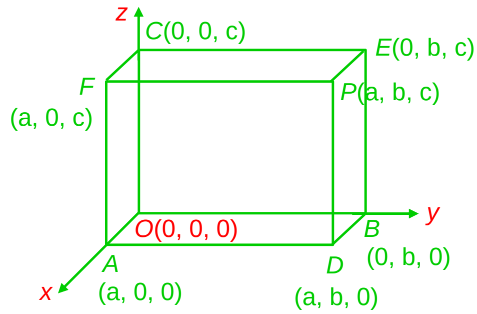
Three dimensional coordinate system is formed by taking three mutually perpendicular planes intersecting at a point O.





The lines XOX′, YOY′ and ZOZ′ at which the three planes intersect are mutually perpendicular to each other and are referred to as the *x*-axis, y-axis and z-axis respectively. The distance measured from XY plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ′ are taken as negative. We specify the coordinates of a point by specifying the *x*, *y* and *z* coordinates of a point.

If the coordinates of *P* are (*a*, *b*, *c*), we mark the coordinates of other points as shown in the figure.



3 DISTANCE BETWEEN TWO POINTS

To find the distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in space.

From the above figure we note that,

$\angle PAN = 90°$

Applying Pythagoras theorem,

$PN^2 = PA^2 + AN^2$

$\angle PNQ = 90°$

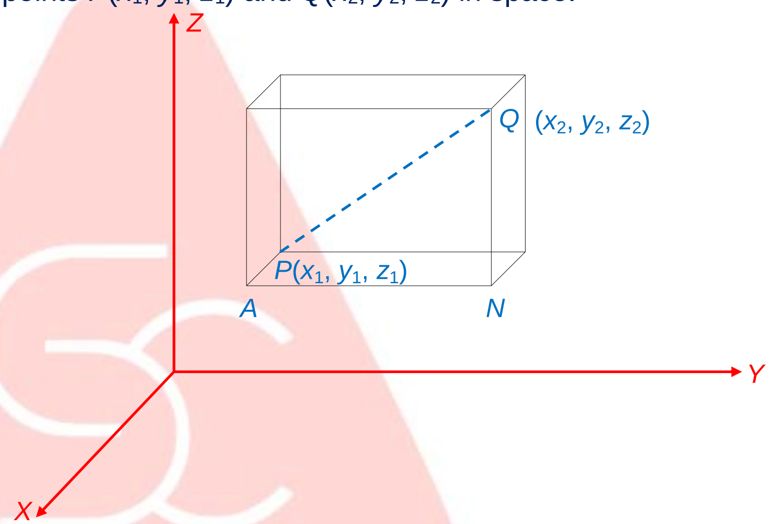
$\therefore\ PQ^2 = PN^2 + NQ^2$

$\therefore\ PQ^2 = PA^2 + AN^2 + NQ^2$

$PA = (x_2 - x_1)$

$AN = (y_2 - y_1)$

$NQ = (z_2 - z_1)$



$$\therefore\ PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$\therefore\ PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

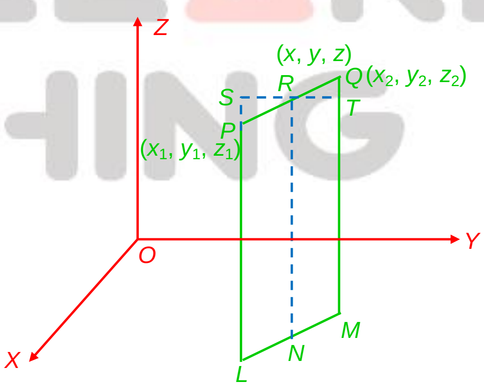
4 SECYION FORMULA

To find the coordinates of a point R which divides PQ internally in the ratio $m : n$, drop perpendicular from P, R and Q on the xy plane. Let the foot of perpendiculars on the xy plane be denoted as L, N and M.

$\because\ \Delta PRS$ is similar to ΔRQT.

$\therefore$ the corresponding sides are proportional.

$$\therefore\ \frac{PR}{RQ} = \frac{m}{n} = \frac{SP}{QT} = \frac{SR}{RT}$$



Now, $SP = (SL - PL) = (RN - PL) = (z - z_1)$

$QT = (QM - TM) = (QM - RN) = (z_2 - z)$

$$\therefore\ \frac{m}{n} = \frac{(z - z_1)}{(z_2 - z)}$$

$$\therefore\ (mz_2 - mz) = nz - nz_1$$

$z(m + n) = mz_2 + nz_1$

$$z = \frac{(mz_2 + nz_1)}{(m+n)}$$

Similarly, $y = \dfrac{(my_2 + ny_1)}{(m+n)}$, $x = \dfrac{(mx_2 + nx_1)}{(m+n)}$

∴ The coordinates of point R are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

Similarly if R divides PQ externally in the ratio of $m : n$, then the coordinates of R are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

Important formulae/points

- *Distance formula* $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
- *Section formula* $x = \dfrac{mx_2 + nx_1}{m+n}, y = \dfrac{my_2 + ny_1}{m+n}, z = \dfrac{mz_2 + nz_1}{m+n}$
- *For internal division m : n is positive.*
- *For external division m : n is negative.*

5 DIRECTION COSINES AND DIRECTION RATIOS

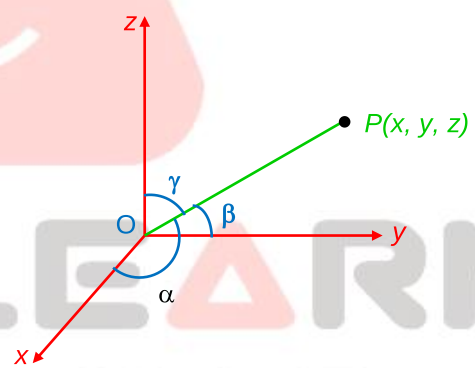
Let α, β, γ be the angle which a line makes with the positive directions of x-axis, y-axis and z-axis respectively.

$l = \cos\alpha$

$m = \cos\beta$

$n = \cos\gamma$

here, l, m and n are referred to as the direction cosines of the line segment.



The numbers a, b, c are referred as the direction ratios such that

$$\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$$

- If l, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$

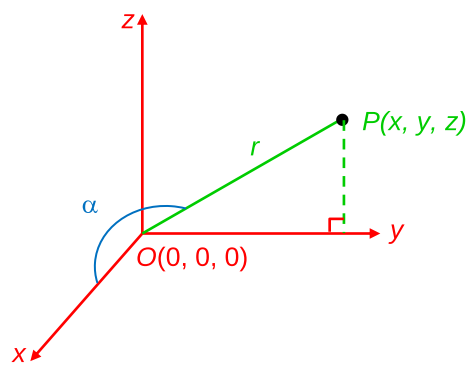
Let OP be drawn parallel to the line with direction cosines l, m and n. Let the length of OP be r. Then $\cos\alpha = \dfrac{x}{r}$

∴ $x = lr$

Similarly $y = mr$

$z = nr$

$x^2 + y^2 + z^2 = r^2$



substituting the value of x, y, z in the above equation

$$l^2 r^2 + m^2 r^2 + n^2 r^2 = r^2 \quad \Rightarrow \quad l^2 + m^2 + n^2 = 1$$

- The direction cosines of a line are l, m and n and direction ratios be a, b, c then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

using the properties of ratio and proportion,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}}$$

$\because \quad l^2 + m^2 + n^2 = 1$

$\therefore \quad l = \pm\dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$

$$m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

5.1 DIRECTION COSINES OF THE LINE JOINING THE POINTS $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

Let the direction cosines of PQ be l, m and n and α be the angle which PQ makes with the positive direction of x-axis.

Projection of PQ on x-axis $= (x_2 - x_1)$

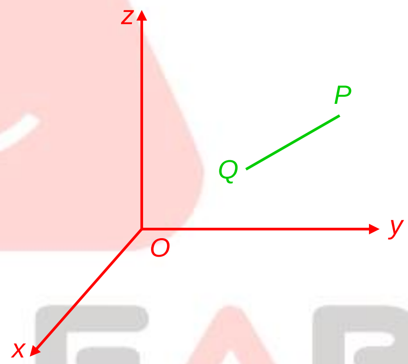
$= PQ \cos\alpha$

$= (PQ) \,.\, l$

Similarly,

Projection of PQ on y-axis $= (y_2 - y_1)$

$= PQ.\ m$



Projection of PQ on z-axis $= (z_2 - z_1)$

$= PQ.\ n$

Equating the values of PQ,

$$\frac{(x_2 - x_1)}{l} = \frac{(y_2 - y_1)}{m} = \frac{(z_2 - z_1)}{n}$$

The direction cosines of the line are

$$\pm\frac{(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}},$$

$$\pm\frac{(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}},$$

$$\pm\frac{(z_2 - z_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

Important formulae/points

- *Direction cosines*

  $l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$

  $l^2 + m^2 + n^2 = 1$

- *Direction ratios* $k\cos\alpha, k\cos\beta, k\cos\gamma$ *; where k is any real number.*