Trigonometric Functions

In an extended way the study is also of the angles forming the elements of a triangle. Logically, a discussion of the properties of a triangle; solving a triangle, physical problems in the area of heights and distances using the properties of a triangle – all constitute a part of the study. It also provides a method of solution of trigonometric equations.

1 MEASUREMENT OF ANGLES

There are two methods used in measuring angles. The first one uses the idea of right angle as a fundamental unit. When two lines intersect, four angles are formed. If the lines intersect in such a way that all the four angles are equal, then each is called a right angle. The other units in this system are:

1 right angle = 90 degrees (= 90°)
1 degree = 60 minutes (= 60')
1 minute = 60 seconds (= 60")

In the second method the fundamental unit is radian, which is the angle subtended at the centre of any circle by an arc of the circle equal in length to the radius.

Though defined with respect to a circle it is totally independent of the circle (its radius or its centre location).

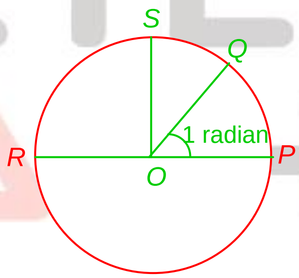
In fact it is a constant angle for otherwise it cannot be chosen as a unit of measurement. The formula of conversion of angles from radian measure to degree measure is

π radians = 180 degrees or $\pi^c = 180°$

conversely from degrees to radians

$1° = (\pi/180)$ of a radian.

It is usual to write π for 180°, $\pi/2$ for 90°, $\pi/4$ for 45° *etc.* It may be mentioned that this means that π radians = 180°; $(\pi/2)$ radians = 90° and so on.



The unit radian is denoted by *c* (circular measure) and it is customary to omit this symbol *c.*

Thus, when an angle is denoted as $\frac{\pi}{2}$, it means that the angle is $\frac{\pi}{2}$ radians where π is the number with approximate value 3.14159.

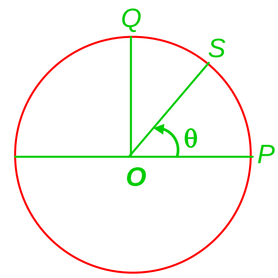
LENGTH OF AN ARC OF A CIRCLE

Consider an arc *PS* of a circle which subtends an angle θ (θ radians). Let $\angle POQ$ = 1 radian.

Then arc $PQ = r$
If the length of the arc $PS = l$, then

$$\frac{\angle POS}{\angle POQ} = \frac{\text{arc } PS}{\text{arc } PQ}$$

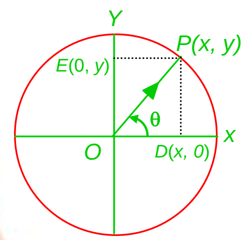
i.e., $\frac{\theta}{1} = \frac{l}{r}$ or $l = r\theta$



2 TRIGONOMETRIC FUNCTIONS OF AN ANGLE

The six trigonometric ratios sine, cosine, tangent, cotangent, secant and cosecant of an angle θ, $0° < \theta < 90°$ are defined as the ratios of two sides of a right-angled triangle with θ as one of the angles. However these can be defined through a unit circle more elegantly.

Draw a unit circle and take any two diameters at right angle as X and Y axes. Taking OX as the initial line, let $\overrightarrow{OP}$ be the radius vector corresponding to an angle θ, where P lies on the unit circle. Let (x, y) be the coordinates of P.



Then by definition :

$\cos\theta = x$, the x coordinate of P

$\sin\theta = y$, the y coordinate of P

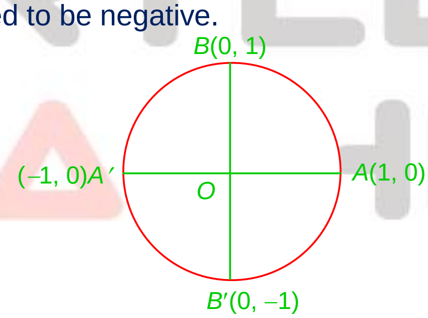
$\tan\theta = \frac{y}{x}, x \neq 0$

$\cot\theta = \frac{x}{y}, y \neq 0$

$\sec\theta = \frac{1}{x}, x \neq 0$

$\text{cosec}\,\theta = \frac{1}{y}, y \neq 0$

Angles measured anticlockwise from the initial line OX are deemed to be positive and angles measured clockwise are considered to be negative.



Since we can associate a unique radius vector $\overrightarrow{OP}$ and a unique point P with each angle θ, we say x and y and their ratios are functions of θ. This justifies the term trigonometric 'function'. This definition holds good for all angles positive, negative, acute or not acute (irrespective of the magnitude of the angle).

This definition also helps us to write the sine and cosine of four important angles 0°, 90°, 180° and 270° easily.

$\theta = 0° \Rightarrow A(1, 0)$
$\theta = 90° \Rightarrow B(0, 1)$
$\theta = 180° \Rightarrow A'(-1, 0)$
$\theta = 270° \Rightarrow B'(0, -1)$

$\left.\begin{array}{l}\cos 0° = 1 \\ \sin 0° = 0\end{array}\right\}$ $\left.\begin{array}{l}\cos 90° = 0 \\ \sin 90° = 1\end{array}\right\}$ $\left.\begin{array}{l}\cos 180° = -1 \\ \sin 180° = 0\end{array}\right\}$ $\left.\begin{array}{l}\cos 270° = 0 \\ \sin 270° = -1\end{array}\right\}$

We can also infer the quadrant rule for sine, cosine and tangent easily.

I quadrant	**II quadrant**	**III quadrant**	**IV quadrant**
sin, cosine and tangent are positive	sine alone is positive	tangent alone is positive	cosine alone is positive

$90° \to$ Point $B(0, 1)$

Since, $\tan\theta = \dfrac{y}{x}, x \neq 0$, $\tan 90° = \dfrac{1}{0}$ and hence undefined. However, as θ increases from 0 to 90°, tanθ increases from 0 to + ∞.

Similarly, sec 90°, cot 0°, cosec 0° are also undefined. 360° and 0° correspond to one and the same point $A(1, 0)$. Therefore, the trigonometric functions of 360° are the same as trigono-metric functions of 0°.

sin 360° = 0, cos 360° = 1 and tan 360° = 0

Since θ, 2π + θ, 4π + θ, 6π + θ, ..., $2n\pi + \theta$ and θ – 2π, θ – 4π, θ – 6π,....., $\theta - 2n\pi$, all correspond to the same radius vector, the trigonometric functions of all these angles are the same as those of θ.

∴ $\sin(2n\pi + \theta) = \sin\theta$ and $\sin(\theta - 2n\pi) = \sin\theta$

$\cos(2n\pi + \theta) = \cos\theta$ and $\cos(\theta - 2n\pi) = \cos\theta$

$\tan(2n\pi + \theta) = \tan\theta$ and $\tan(\theta - 2n\pi) = \tan\theta$

The range of the trigonometric ratios in the four quadrants are depicted in the following table.

Y

In the second quadrant

sine	decreases from	1 to 0
cosine	decreases from	0 to –1
tangent	increases from	–∞ to 0
cotangent	decreases from	0 to – ∞
secant	increases from	–∞ to – 1
cosecant	increases from	1 to ∞

In the first quadrant

sine	increases from	0 to 1
cosine	decreases from	1 to 0
tangent	increases from	0 to ∞
cotangent	decreases from	∞ to 0
secant	increases from	1 to ∞
cosecant	decreases from	∞ to 1

x′ O X

In the third quadrant

sine	decreases from	0 to –1
cosine	increases from	–1 to 0
tangent	increases from	0 to ∞
cotangent	decreases from	∞ to 0
secant	decreases from	–1 to –∞
cosecant	increases from	–∞ to –1

In the fourth quadrant

sine	increases from	–1 to 0
cosine	increases from	0 to 1
tangent	increases from	–∞ to 0
cotangent	decreases from	0 to –∞
secant	decreases from	∞ to 1
cosecant	decreases from	–1 to –∞

Y′

2.1 TRIGONOMETRIC FUNCTIONS OF (–θ)

Let *OP* and *OP′* be the radii vectors, on the unit circle corresponding to θ and –θ. If (x, y) are the coordinates of *P,* then $(x, -y)$ would be the coordinates of *P′*.

Now sinθ = y and sin(–θ) = $-y$

Hence, sin(–θ) = –sinθ

Similarly, cos(–θ) = cosθ and tan(–θ) = –tanθ

2.2 CIRCULAR FUNCTIONS OF ALLIED ANGLES

When θ is an acute angle, 90° – θ is called the angle complementary to θ. Trigonometric functions of 90° – θ are related to trigonometric functions of θ as follows :

sin(90° – θ) = cosθ	cosec(90° – θ) = secθ
cos(90° – θ) = sinθ	sec(90° – θ) = cosecθ
tan(90° – θ) = cotθ	cot(90° – θ) = tanθ

When θ is acute, θ and $180° - \theta$ are called supplementary angles.

$\sin(180° - \theta) = \sin\theta$ $\quad$ $\text{cosec}(180° - \theta) = \text{cosec}\theta$

$\cos(180° - \theta) = -\cos\theta$ $\quad$ $\sec(180° - \theta) = -\sec\theta$

$\tan(180° - \theta) = -\tan\theta$ $\quad$ $\cot(180° - \theta) = -\cot\theta$

Formulae for functions of :$180° + \theta$, $270° - \theta$, $270° + \theta$, $360° - \theta$ can all be derived with the help of unit circle definition.

There is an easy way to remember these formulae. First of all think of θ as an acute angle. Angles like $180° \pm \theta$, $360° \pm \theta$, $-\theta$ can be considered as angles associated with the horizontal line, angles like $90° - \theta$, $90° + \theta$, $270° \mp \theta$ can be considered as angles associated with vertical line. When associated with the horizontal line, the magnitude of the function does not change, whereas when associated with the vertical line the function changes to the corresponding complementary value. For example $\sin(180° + \theta)$ will be only $\sin\theta$ (in magnitude) plus or minus and $\cos(180° - \theta)$ will be cosine θ only in magnitude.

To decide upon the sign, consider the quadrant in which the angle falls and decide the sign by the quadrant rule.

For example, $\sin(180° + \theta)$ is $\sin\theta$ (in magnitude) $(180° + \theta)$ lies in third quadrant and hence $\sin(180° + \theta)$ is negative.

$\therefore \quad \sin(180° + \theta) = -\sin\theta$

Again consider $\cos(360° - \theta)$: first of all, it should be $\cos\theta$ (in magnitude); since $(360° - \theta)$ lies in IV quadrant, its cosine is positive.

$\therefore \quad \cos(360° - \theta) = \cos\theta$.

Again consider $\tan(90° + \theta)$: This should be $\cot\theta$ and must have a negative sign since $(90° + \theta)$ is in II quadrant and hence $\tan(90° + \theta)$ is negative.

$\therefore \quad \tan(90° + \theta) = -\cot\theta$

TABLE OF FORMULAE FOR ALLIED ANGLES

	$180° - \theta$	$180° + \theta$	$360° - \theta$	$-\theta$	$90° - \theta$	$90° + \theta$	$270° - \theta$	$270° + \theta$
sin	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$-\cos\theta$	$-\cos\theta$
cos	$-\cos\theta$	$-\cos\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
tan	$-\tan\theta$	$\tan\theta$	$-\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$\cot\theta$	$-\cot\theta$

These formulae are not memorized but derived as and when the occasion demands according to the rule explained above.

Trigonometric ratios of 30°, 45° and 60° are of great importance in solving problems on heights and distances. These along with 0° and 90° are written in tabular form and remembered.

ANGLE / RATIO	0°	30°	45°	60°	90°
sine	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cosine	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
tangent	0	$1/\sqrt{3}$	1	$\sqrt{3}$	undefined
cotangent	undefined	$\sqrt{3}$	1	$1/\sqrt{3}$	0
secant	1	$2/\sqrt{3}$	$\sqrt{2}$	2	undefined
cosecant	undefined	2	$\sqrt{2}$	$2/\sqrt{3}$	1

2.3 SOME IMPORTANT FACTS

The following may be noted

(i) For any power n, $(\sin A)^n$ is written as $\sin^n A$. Similarly for all other trigonometric ratios.
(ii) cosec$A$, sec$A$ and cot$A$ are respectively the reciprocals of sin$A$, cos$A$ and tan$A$.
(iii) $\sin^2 A + \cos^2 A = 1$; $1 + \tan^2 A = \sec^2 A$ and $1 + \cot^2 A = \text{cosec}^2 A$.
(iv) sec$A$ – tan$A$ and sec$A$ + tan$A$ are reciprocals. So also are cosec$A$ – cot$A$ and cosec$A$ + cot$A$.

Whenever sec$A$ or tan$A$ is thought of for an angle A, it is necessary to stress that, $A \neq \pi/2$ particularly, and generally $A \neq n\pi + \frac{\pi}{2} (n \in \boldsymbol{N},$ where $\boldsymbol{N}$ is the set of natural numbers).

(v) $|\sin A| \leq 1 \Rightarrow -1 \leq \sin A \leq 1$
$|\cos A| \leq 1 \Rightarrow -1 \leq \cos A \leq 1$
$|\text{cosec}A| \geq 1 \Rightarrow \text{cosec}A \geq 1$ or $\text{cosec}\,A \leq -1$
$|\sec A| \geq 1 \Rightarrow \sec A \geq 1$ or $\sec A \leq -1$

$$\sin\left(\frac{\pi}{2} - A\right) = \sin\left(\frac{\pi}{2} + A\right) = \cos A$$

(vi) $$\cos\left(\frac{\pi}{2} - A\right) = -\cos\left(\frac{\pi}{2} + A\right) = \sin A$$

$\sin(\pi - A) = -\sin(\pi + A) = \sin A$
$\cos(\pi - A) = \cos(\pi + A) = -\cos A$
$\tan(\pi - A) = -\tan(\pi + A) = -\tan A$

(vii) The trigonometric ratios are also called as trigonometric functions. They are also sometimes called circular functions.

The trigonometric functions, apart from possessing many other properties exhibit a property of the values being repeated when the angle is changed (increased or decreased) by a constant value. Such a property is referred to as periodicity.

Thus $\sin x = \sin(x + 2\pi) = \sin(x + 4\pi)$
$= \sin(x - 2\pi) = \sin(x + 2k\pi)$, k an integer.
$\cos x = \cos(x + 2\pi) = \cos(x + 4\pi)$
$= \cos(x - 2\pi) = \cos(x + 2k\pi)$, k an integer.

Hence both sin$x$ and cos$x$ are periodic functions of period 2π radians. From (v), it is clear that they are also bounded functions.

cosec x and sec x, whenever they exist, are also periodic of period 2π radians. tan x and cot x, when they exist, are periodic of period π radians.

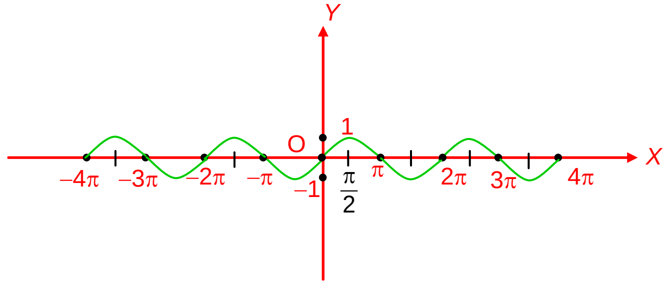
tan x, sec x, cosec x and cot x are unbounded functions.

2.4 GRAPH OF TRIGONOMETRIC FUNCTIONS

Graph of $y = \sin x$

From the knowledge of trigonometry we compute the following table :

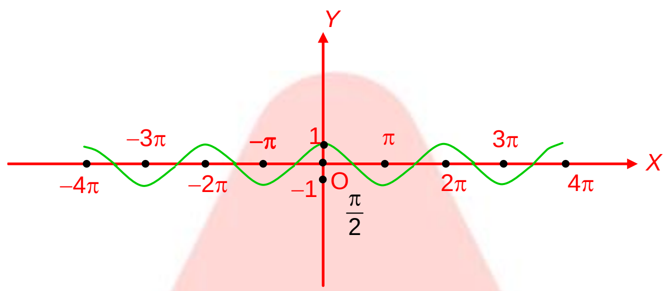
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
sin$x$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0



$y = \sin x$

Graph of $y = \cos x$

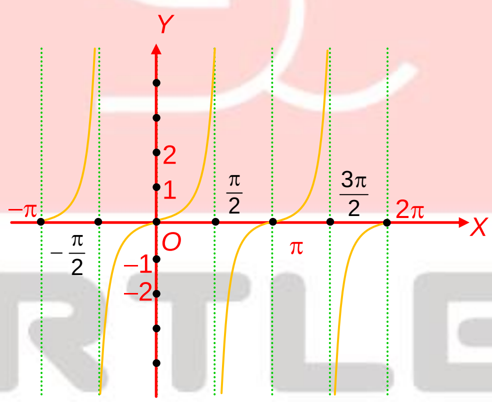
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1



$y = \cos x$

Graph of $y = \tan x$

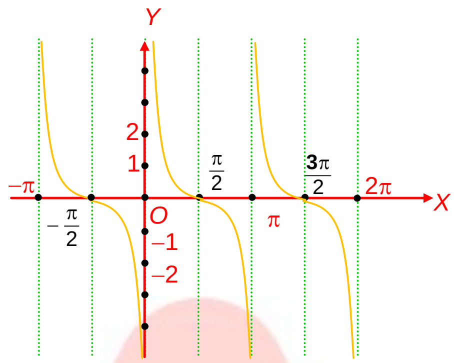
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0



$y = \tan x$

Graph of $y = \cot x$

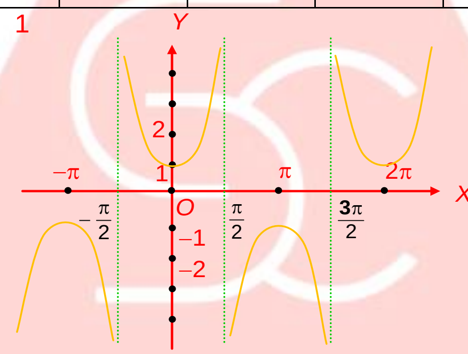
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cot x$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	undefined



y = cot x

Graph of $y = \sec x$

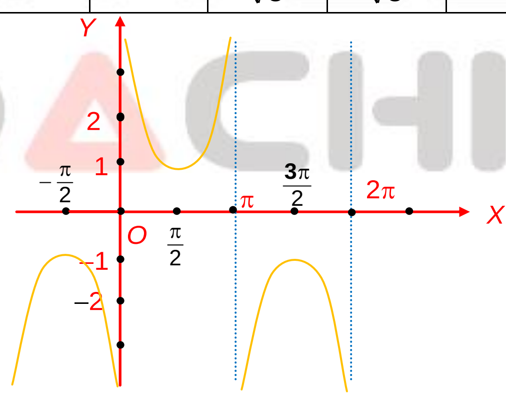
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
sec$x$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	−2	$-\sqrt{2}$	$\frac{-2}{\sqrt{3}}$	−1



y = sec x

Graph of $y = \text{cosec } x$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cosec$x$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined



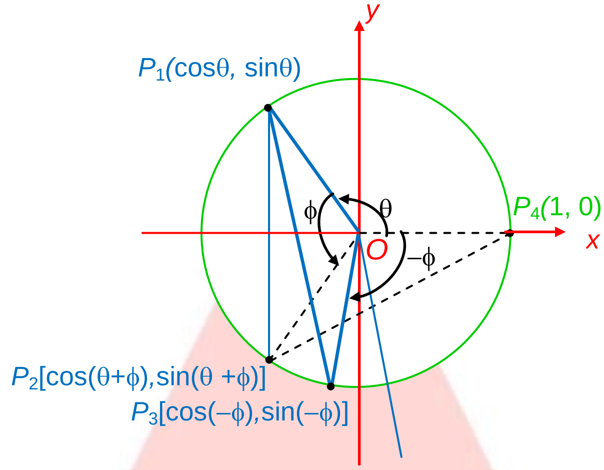
y = cosec x

3 CIRCULAR FUNCTIONS OF COMPOUND ANGLES

An equation involving trigonometric functions, which is true for all those values of θ for which the functions are defined is called a trigonometric identity, otherwise it is a trigonometric equation.

We shall now derive some results which are useful in simplifying trigonometric equations. *To* *prove:*

$\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$ ...(i)



Consider a unit circle with origin as its centre.

Let $\angle P_4OP_1 = \theta$ and $\angle P_1OP_2 = \phi$

$\therefore\ \angle P_4OP_2 = \theta + \phi$

$\angle P_4OP_3 = -\phi$

Coordinate of P_1, P_2, P_3, P_4 are

$P_1(\cos\theta, \sin\theta)$

$P_2[\cos(\theta + \phi), \sin(\theta + \phi)]$

$P_3[\cos(-\phi), \sin(-\phi)]$

$P_4(1, 0)$

$\Delta P_1O P_3$ is congruent to $\Delta P_2O P_4$

$\because\ OP_1 = OP_4 = OP_3 = OP_2$ = Radius of the circle

$\angle P_1OP_3 = \angle P_2OP_4 = 360 - (\theta + \phi)$

$\therefore$ By Side Angle Side, the triangles are congruent.

$\therefore\ P_1P_3 = P_2P_4$

Applying the distance formula,

$P_1P_3^2 = [\cos\theta - \cos(-\phi)]^2 + [\sin\theta - \sin(-\phi)]^2$

$= (\cos\theta - \cos\phi)^2 + (\sin\theta + \sin\phi)^2$

[using $\cos(-\phi) = \cos\phi$ and $\sin(-\phi) = -\sin\phi$]

$= \cos^2\theta + \cos^2\phi - 2\cos\theta \cos\phi + \sin^2\theta + \sin^2\phi + 2\sin\theta \sin\phi$

$= 2 - 2(\cos\theta\ \cos\phi - \sin\theta \sin\phi)$

$P_2P_4^2 = [1 - \cos(\theta + \phi)]^2 + [0 - \sin(\theta + \phi)]^2$

$= 1 - 2\cos(\theta + \phi) + \cos^2(\theta + \phi) + \sin^2(\theta + \phi)$

$= 2 - 2\cos(\theta + \phi)$.

Since $P_1P_3 = P_2P_4$, we have $P_1P_3^2 = P_2P_4^2$.

$\therefore\ 2 - 2(\cos\theta\ \cos\phi - \sin\theta \sin\phi) = 2 - 2\cos(\theta + \phi)$

Hence $\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$

$\cos(\theta - \phi) = \cos\theta \cos\phi + \sin\theta \sin\phi$ ...(ii)

Replacing ϕ by $-\phi$ in identity (i), we get

$\cos(\theta - \phi) = \cos\theta \cos(-\phi) - \sin\theta \sin(-\phi)$

or $\cos(\theta - \phi) = \cos\theta \cos\phi + \sin\theta \sin\phi$

3.1 ADDITION FORMULAE

$$\left.\begin{aligned}\sin(A+B)&=\sin A\cos B+\cos A\sin B\\ \cos(A+B)&=\cos A\cos B-\sin A\sin B\\ \tan(A+B)&=\frac{\tan A+\tan B}{1-\tan A\tan B}\end{aligned}\right\}\qquad ...(i)$$

3.2 SUBTRACTION FORMULAE

$$\left.\begin{aligned}\sin(A-B)&=\sin A\cos B-\cos A\sin B\\ \cos(A-B)&=\cos A\cos B+\sin A\sin B\\ \tan(A-B)&=\frac{\tan A-\tan B}{1+\tan A\tan B}\end{aligned}\right\}\qquad ...(ii)$$

These formulae can be derived with the help of elementary geometry. The subtraction formulae can be derived from the addition formulae by replacing B by $-B$.

Function of $(A + B + C)$

1. $\sin(A + B + C) = \Sigma \sin A \cos B \cos C - \sin A \sin B \sin C$

   Proof : $\sin\{A+ (B + C)\} = \sin A \cdot \cos(B + C) + \cos A \cdot \sin(B + C)$

   $= \sin A\{\cos B \cos C - \sin B \sin C\} + \cos A \{\sin B \cos C + \cos B \sin C\}$

   $= \Sigma \sin A \cos B \cos C - \sin A \sin B \sin C$ ...(iii)

2. $\cos(A + B + C) = \cos A \cos B \cos C - \Sigma \cos A \sin B \sin C$

   Proof is similar to that of the previous formula.

3. $\tan (A + B + C) = \dfrac{\Sigma \tan A - \tan A \tan B \tan C}{1-\Sigma \tan B \tan C}$.

   Proof: $\tan(A + B + C) = \tan\{A +(B + C)\} = \dfrac{\tan A+\tan(B+C)}{1-\tan A.\tan(B+C)} = \dfrac{\tan A+\dfrac{\tan B+\tan C}{1-\tan B\tan C}}{1-\tan A.\left(\dfrac{\tan B+\tan C}{1-\tan B\tan C}\right)}$

$$=\frac{(\tan A+\tan B+\tan C)-\tan A.\tan B.\tan C}{1-(\tan A\tan B+\tan B\tan C+\tan C\tan A)}=\frac{S_1-S_3}{1-S_2}$$

where S_r = sum of the products (of tangents) taken r at a time.

3.4 MULTIPLE ANGLE FORMULAE

3.4.1 Functions of 2A

(i) $\sin 2A = 2\sin A \cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

These are special cases of the addition formulae by taking $B = A$. The formulae for $\cos 2A$ leads to two results whose application occurs often in problems.

$1 + \cos 2A = 2\cos^2 A$ and $1 - \cos 2A = 2\sin^2 A$

Besides these, $\sin 2A=\dfrac{2\tan A}{1+\tan^2 A}$ and $\cos 2A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$

If A be replaced by $\dfrac{A}{2}$, these formulae reduce to

(i) $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2}$

(ii) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

$= 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}$

(iii) $\tan A = = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

3.4.2 Functions of 3A

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

Proof: $\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$

$= 2\sin A \cos^2 A + \sin A\,(1 - 2\sin^2 A)$

$= 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

Proof : $\cos 3A = \cos(2A + A) = \cos 2A \,.\, \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - 2\sin^2 A \cos A$

$= (2\cos^2 A - 1)\cos A - 2\cos A\,(1 - \cos^2 A)$

$= 4\cos^3 A - 3\cos A$

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Proof: $\tan 3A = \dfrac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \dfrac{\dfrac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \dfrac{2\tan^2 A}{1-\tan^2 A}} = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Note: This formula can be derived from the expansion of $\tan(A + B + C) = \dfrac{S_1 - S_3}{1 - S_2}$ by taking $B = C = A$.

3.5 EXPRESSING PRODUCTS OF TRIGONOMETRIC FUNCTIONS AS SUM OR DIFFERENCE

(i) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$
(ii) $2\cos A \sin B = \sin(A + B) - \sin(A - B)$
(iii) $2\cos A \cos B = \cos(A + B) + \cos(A - B)$ ...(i)
(iv) $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

The above four formulae can be obtained by expanding the right hand side and simplifying.

Note : In the fourth formula, there is a change in the pattern. Angle $(A - B)$ comes first and $(A + B)$ later. In the first quadrant, the greater the angle, the less the cosine. Hence cosine of the smaller angle is written first [to get a positive result].

3.7 SOME MORE RESULTS

- $\sin(A + B) \times \sin(A - B) = \sin^2 A - \sin^2 B$
- $\cos(A + B) \times \cos(A - B) = \cos^2 A - \sin^2 B$
- $\sin 18° = \dfrac{(\sqrt{5} - 1)}{4} = \cos 72°$
- $\cos 18° = \dfrac{\sqrt{10 + 2\sqrt{5}}}{4} = \sin 72°$

- $\cos 36^\circ = \frac{(\sqrt{5}+1)}{4} = \sin 54^\circ$
- $\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54^\circ$
- $\tan 22\frac{1}{2}^\circ = (\sqrt{2}-1)$

Triple Angle Formulae

- $\sin\theta \sin(60^\circ - \theta) \sin(60^\circ + \theta) = \frac{1}{4}\sin 3\theta$
- $\cos\theta \cos(60^\circ - \theta) \cos(60^\circ + \theta) = \frac{1}{4}\cos 3\theta$
- $\tan\theta \tan(60^\circ - \theta) \tan(60^\circ + \theta) = \tan 3\theta$

4 CONDITIONAL TRIGONOMETRICAL IDENTITIES

4.1 IDENTITIES

A trigonometric equation is an identity if it is true for all values of the angle or angles involved.

4.2 CONDITIONAL IDENTITIES

When the angles involved satisfy a given relation, the identity is called conditional identity. In proving these identities we require properties of complementary and supplementary angles.

4.3 SOME IMPORTANT CONDITIONAL IDENTITIES

If $A + B + C = \pi$, then

- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
- $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$
- $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
- $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$
- $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$
- $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$
- $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}.\cot\frac{B}{2}\cot\frac{C}{2}$

4.4 SOME OTHER USEFUL RESULTS

1. $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots\ldots\ldots\ldots$ to n terms $= \frac{\sin[\alpha + (n-1)\beta/2]\sin(n\beta/2)}{\sin(\beta/2)}$
2. $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots\ldots\ldots\ldots$ to n terms
   $= \frac{\cos[\alpha + (n-1)\beta/2]\sin(n\beta/2)}{\sin(\beta/2)}$.
3. $\sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A}$
4. $\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1-\sin A}$

5. $\sin A \pm \cos A = \sqrt{2}\sin\left(\frac{\pi}{4} \pm A\right) = \sqrt{2}\cos\left(A \mp \frac{\pi}{4}\right)$

6. $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$

7. $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$

8. If A, B, C are the angles of a triangle (or $A + B + C = \pi$), then the result (1) implies $\Sigma \sin A \cos B \cos C = \Pi \sin A$.
   Result (2) $\Rightarrow \Sigma \cos A \sin B \sin C = 1 + \Pi \cos A$
   Result (3) $\Rightarrow \Sigma \tan A = \Pi \tan A$
   Putting $\alpha = A$, $\beta = B$ and $\gamma = C$ in result (9) and (10),
   we get $\Sigma \cos A = 1 + 4\Pi \cos\frac{A}{2}$
   and $\Sigma \sin A = 4\Pi \cos\frac{A}{2}$ respectively.
   Putting $A/2$, $B/2$, $C/2$ in place of A, B, C respectively in result (3), and utilizing $A + B + C = \pi$,
   we get,
   $$\Sigma \tan\frac{B}{2}\tan\frac{C}{2} = 1.$$
   For a triangle we can also show that $\Sigma \cot B \cot C = 1$; $\Sigma \cot\frac{A}{2} = \Pi \cot\frac{A}{2}$;

- $\Sigma \sin 2A = 4\Pi \sin A$; $\Sigma \cos 2A = -1 - 4\Pi \cos A$; $\Sigma \cos^2 A = -1 - 2\Pi \cos A$, *etc*.

5 BOUNDS OF THE EXPRESSION: $a\cos\theta + b\sin\theta$

$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right)$

$= \sqrt{a^2+b^2}\,(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$

$= \sqrt{a^2+b^2}\,\sin(\theta+\alpha)$, where $\tan\alpha = \frac{a}{b}$

Also, $a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\,\cos(\theta-\beta)$, where $\tan\beta = \frac{b}{a}$

$-1 \le \sin(\theta+\alpha) \le 1$

and hence, $-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$

Thus the expression $a\cos\theta + b\sin\theta$ is bounded above by $\sqrt{a^2+b^2}$ and bounded below by $-\sqrt{a^2+b^2}$.

MIND MAP

TRIGONOMETRIC FUNCTIONS

Measurement of Angle

1. π radians $\equiv$ 180 degrees
   $1^c \equiv 57°\ 17'\ 44.8''$
2. 1 right angle = 90°
   $1° \equiv 60'$ and $1' = 60''$
3. Angle (in radians) = $\frac{\text{Arc}}{\text{Radius}}$

Domain and Range

Function	Domain	Range
1. $y = \sin x$	$\boldsymbol{R}$	$[-1, 1]$
2. $y = \cos x$	$\boldsymbol{R}$	$[-1, 1]$
3. $y = \tan x$	$\boldsymbol{R}-(2n+1)\frac{\pi}{2}$	$\boldsymbol{R}$
4. $y = \text{cosec}\,x$	$\boldsymbol{R}-n\pi$	$(-\infty, -1] \cup [1, \infty)$
5. $y = \sec x$	$\boldsymbol{R}-(2n+1)\frac{\pi}{2}$	$(-\infty, -1] \cup [1, \infty)$
6. $y = \cot x$	$\boldsymbol{R} - n\pi$	$\boldsymbol{R}$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
$\tan(A \pm B) = \frac{\tan \pm \tan B}{1 \mp \tan A \tan B}$

$2\sin A\cos B = \sin(A + B) + \sin(A - B)$
$2\cos A\sin B = \sin(A + B) - \sin(A - B)$
$2\cos A\cos B = \cos(A + B) + \cos(A - B)$
$2\sin A\sin B = \cos(A - B) - \cos(A + B)$

All t-ratios are positive in Q_1
Only sin and cosec are positive in Q_2
Only tan and cot are positive in Q_3
Only cos and sec are positive in Q_4

$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$
$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$
$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$
$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

$-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$
for all values of θ

$\tan(A + B + C) = \frac{\Sigma\tan A - \Pi\tan A}{1-\Sigma\tan A\tan B}$
$\sin(A + B + C) = \Sigma\sin A\cos B\cos C - \Pi\sin A$
$\cos(A + B + C) = \Pi\cos A - \Sigma\cos A\sin B\sin C$
$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

$\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
$\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

For any three angles α, β, γ
$\Sigma\cos\alpha + \cos(\Sigma\alpha) = 4\Pi\cos\frac{\alpha+\beta}{2}$
$\Sigma\sin\alpha - \sin(\Sigma\alpha) = 4\Pi\sin\frac{\alpha+\beta}{2}$

$4\sin(60° - \theta)\sin\theta\sin(60° + \theta) = \sin 3\theta$
$4\cos(60° - \theta)\cos\theta\cos(60° + \theta) = \cos 3\theta$
$\tan(60° - \theta)\tan\theta\tan(60° + \theta) = \tan 3\theta$

$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$
$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$
$1 + \cos 2\theta = 2\cos^2\theta$; $1 - \cos 2\theta = 2\sin^2\theta$;
$\tan^2\theta/2 = \frac{1-\cos\theta}{1+\cos\theta}$
$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

Conditional Identities

If $A + B + C = \pi$
1. $\Sigma\tan A = \Pi\tan A$, $\Sigma\tan A/2.\ \tan B/2 = 1$
2. $\Sigma\cos A = 1 + 4\Pi\sin\frac{A}{2}$
3. $\Sigma\sin A = 4\Pi\cos\frac{A}{2}$
4. $\Sigma\cot A\cot B = 1$, $\Sigma\cot\frac{A}{2} = \Pi\cot\frac{A}{2}$