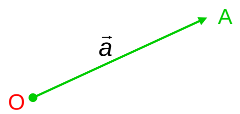
VECTOR ALGEBRA

1 VECTORS AND SCALARS

Physical quantities, like displacements, velocity, acceleration etc. which have not only a value to signify how much (quantity wise) they are; – this aspect is called the magnitude, – but in addition have a directional element in them, are known as vectors.

Any segment *AB* of a straight line which has distinct end points *A* and *B* is called a directed line segment. *A* and *B* are called respectively the initial point and end point of the line segment. *A* vector may be defined in terms of such a line segment and we write $\vec{a} = \overrightarrow{AB}$.



Vectors are generally printed in bold faced type (in printing). When writing in the manuscript form the notation $\vec{a}$, $\overrightarrow{AB}$ may be used.

A vector, thus, is geometrically represented by a directed line segment. It is said that $\vec{a}$ is equivalently $\overrightarrow{OA}$ and they are vectorially indistinguishable.

2 LAW OF ADDITION OF VECTORS

The laws, governing the algebra of vectors, are so designed as to be particularly applicable in cases of physical quantities like velocity, acceleration, etc.

Thus two vectors $\vec{a}$ and $\vec{b}$ are added according to the parallelogram law of addition; namely:

If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by two, line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$, their sum $\vec{c} = \vec{a} + \vec{b}$, is represented by the diagonal $\overrightarrow{OC}$ of the completed parallelogram OACB.

Sometimes this is also referred to as the triangle law of addition.

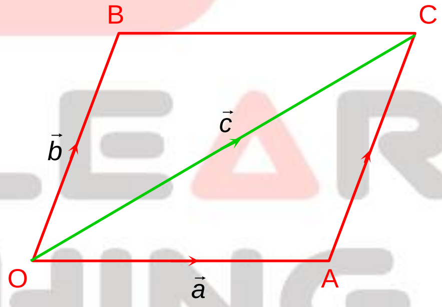
The parallelogram law of addition is

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$$

$$(\vec{a} + \vec{b} = \vec{c})$$

The triangle law is

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$



$$(\vec{a} + \vec{b} = \vec{c})$$

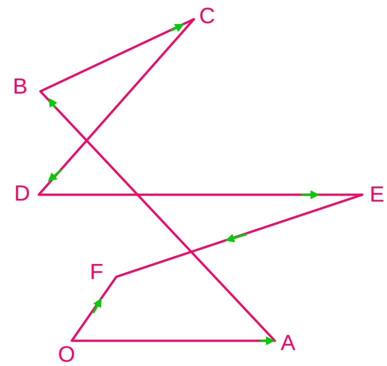
This addition operation, cumulatively, may be had for more than two vectors; and we have

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) = \vec{a} + \vec{b} + \vec{c}$$

For addition of more than two vectors we have a polygon laws of vectors addition which is just an extension of triangle law.

$$\overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EF}=\overrightarrow{OF}$$

As a result if terminus of last vector coincides with the initial point of the first vector, then the sum of vectors is a null vector (a vector with zero magnitude).



3 MAGNITUDE AND MODULUS OF A VECTOR

The magnitude or modulus of a vector $\vec{a}$ refers to its absolute value and is denoted by $|\vec{a}|$. A vector whose modulus is one unit is called a unit vector, and a vector whose modulus is zero is called a zero vector or a null vector. Such a vector has its length zero, and is therefore geometrically represented by a pair of coincident points.

For two vectors $\vec{a}$ and $\vec{b}$, $||\vec{a}| - |\vec{b}|| \le |\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$.

4 MULTIPLICATION OF A VECTOR BY A SCALAR

When a vector is multiplied by a scalar number, its magnitude gets multiplied but direction wise there is no change. Thus $k\vec{a}$ is a vector in the same direction of $\vec{a}$ but magnitude made k times. Thus if in the direction of $\vec{a}$, a unit vector is usually represented as $\hat{a}$ then $\vec{a} = |\vec{a}|\hat{a}$.

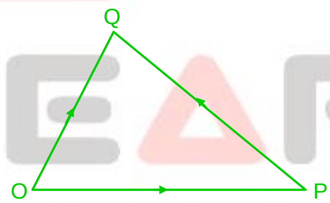
Thus any vector = (its magnitude) unit vector in that direction. It may be also said that $\vec{a}$ and $\hat{a}$ which are direction wise same, are collinear.

5 POSITION VECTOR OF A POINT

The position vector $\vec{r}$ of any point P with respect to the origin of reference O is a vector $\overrightarrow{OP}$.

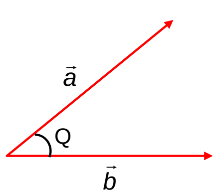
For any two points P and Q in the space, the vector $\overrightarrow{PQ}$ can be expressed in terms of their position vectors (p.v.) as

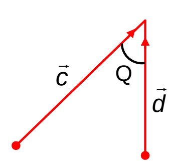
$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$$



6 ANGLE BETWEEN TWO VECTORS

It is defined as the smaller angle formed when the initial points or the terminal points of two vectors are brought together. Angle between two vectors lies in the interval [0, π].



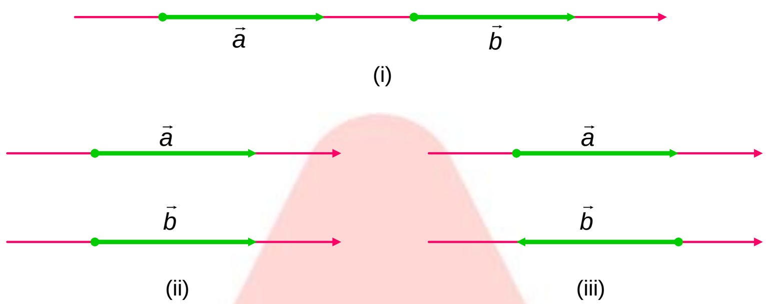


7 PARALLEL VECTORS

Two vectors are parallel if they have same direction. They are also known as like vectors. Non-parallel vectors are known as unlike vectors.

8 COLLINEAR VECTORS

Two vectors $\vec{a}$ and $\vec{b}$ are said to be collinear if they are supported on same or parallel lines. Here if their line of support is parallel, vectors may be parallel or antiparallel. i.e., $\vec{a}$ and $\vec{b}$ can be geometrically represented as



For such vectors $\vec{b} = \lambda\vec{a}$ for some constant λ.

9 COPLANAR VECTORS

A set of vectors is said to be coplanar if they lie in same plane, or are all parallel to the same plane. Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar if their exist a relationship of the form $\vec{c} = \lambda_1 a + \lambda_2 \vec{b}$ for some scalars λ_1 and λ_2.

Any two non-zero vectors which are non-collinear will constitute a plane. Their sum or difference also lies in the same plane.

10 LINEAR COMBINATION OF VECTORS

The vector $r = \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + ... + \alpha_n\vec{a}_n$ where $\alpha_1, \alpha_2, ...\alpha_n$ are any scalars (not all zero), is called linear combination of vectors $\vec{a}_1$, a_2, ... $\vec{a}_n$.

The following results are useful in determining coplanar and collinear vectors.

10.1 FUNDAMENTAL THEOREM IN TWO DIMENSION

If $\vec{a}$ and $\vec{b}$ are non-collinear vectors then any vector $\vec{r}$, coplanar with $\vec{a}$ and $\vec{b}$, can be uniquely expressed as linear combination of $\vec{a}$ and $\vec{b}$, i.e., their exist unique $x, y \in R$ such that $\vec{r} = x\vec{a} + y\vec{b}$.

Also for any two non-collinear vector $\vec{a}$ and $\vec{b}$

$$x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b}$$

$\Rightarrow$ $x = x'$ and $y = y'$

10.2 FUNDAMENTAL THEOREM IN THREE DIMENSION

If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors in space then any vector $\vec{r}$ can be uniquely expressed as linear combination of $\vec{a}, \vec{b}, \vec{c}$.

That is, there exists unique $x, y, z \in R$ such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$.

Also for three non–coplanar vectors $\vec{a}, \vec{b}, \vec{c}$, $x\vec{a} + y\vec{b} + z\vec{c} = x'\vec{a} + y'\vec{b} + z'\vec{c}$

$\Rightarrow$ $x = x'$, $y = y'$ and $z = z'$

10.3 LINEARLY DEPENDENT AND INDEPENDENT VECTORS

A system of vectors $\vec{a}_1, \vec{a}_2 ...., \vec{a}_n$ is said to be linearly dependent if there exist a system of scalars $\lambda_1, \lambda_2, ...\lambda_n$ (not all zero) such that

$$\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + ... + \lambda_n\vec{a}_n = \vec{0}$$

They are linearly independent if every relation of the type $\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + ... + \lambda_n\vec{a}_n = 0$ implies that $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$.

10.4 TEST FOR COLLINEARITY

Three points *A*, *B*, *C* with position vectors $\vec{a}, \vec{b}, \vec{c}$ are collinear if and only if there exist scalars *x*, *y* and *z* not all zero, such that $x\vec{a} + y\vec{b} + z\vec{c} = 0$ where $x + y + z = 0$

Four points *A*, *B*, *C*, *D* with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar if and only if there exist scalars *x*, *y*, *z* and *w*, not all zero, such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ where $x + y + z + w = 0$.

11 ORTHOGONAL SYSTEM OF VECTORS

From the fundamental theorem in three dimensions, any vector can be expressed as linear combination of three non zero non-coplanar vectors. In orthogonal system of vectors we choose these vectors as three mutually perpendicular unit vectors can be denoted as $\hat{i}$, $\hat{j}$ and $\hat{k}$, directed along positive directions of x-axis, y-axis, z-axis respectively.

Any point *P*(*x*, *y*, *z*) has position vector expressed as

$$\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$$

where *x*, *y*, *z* are the scalars representing components of vector $\overrightarrow{OP}$ in the respective directions.

Magnitude of $\overrightarrow{OP} = |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$ represents the distance of P from origin.

Unit vector $$\hat{OP} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Thus $$\hat{OP} = l\hat{i} + m\hat{j} + n\hat{k}$$

where *l*, *m*, *n* are direction cosines of angles α, β, γ made by $\overrightarrow{OP}$ with three directions (called as axes) $\hat{i}$, $\hat{j}$, $\hat{k}$, such that

$$l = \cos\alpha = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$m = \cos\beta = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$n = \cos\gamma = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

12 SOME USEFUL RESULTS

A set of three mutually perpendicular axes *OX*, *OY*, *OZ* along which unit vectors are respectively $\hat{i}$, $\hat{j}$, $\hat{k}$ constitute a reference frame. If a point *P* is represented in its position by the ordered triplet (*x*, *y*, *z*), then the vector $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ represents the position vector of P with respect to the origin O. Therefore *A* (x_1, y_1, z_1) can be represented uniquely by its position vector drawn from O to *A* so that $\overrightarrow{OA} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, where (x_1, y_1, z_1) are the coordinates of A.

A number of useful results, analogous to what are available in coordinate Geometry, (or in complex number theory) is available in vectors also.

Thus if $\vec{b}$ and $\vec{a}$ are the position vectors of *A* and *B*, then the midpoint of *AB* has position vector $\frac{\vec{a} + \vec{b}}{2}$.

More generally a point *P* dividing the join of $\vec{a}$ and $\vec{b}$ in the ratio *m* : *n* has position vector

$\frac{m\vec{b}+n\vec{a}}{m+n}$.

We also have the following:

If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices of a triangle *ABC*, then

(i) the centroid of the triangle has position vector $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$

(ii) the orthocentre O, of the triangle has position vector

$$\frac{(a\sec A)\vec{a}+(b\sec B)\vec{b}+(c\sec C)\vec{c}}{a\sec A+b\sec B+c\sec C}$$

(a, b, c are the sides and A, B, C are the angles of the triangle.)

(iii) The circumcentre S of the triangle has position vector $\frac{\sin 2A\,\vec{a}+\sin 2B\,\vec{b}+\sin 2C\,\vec{c}}{\sin 2A+\sin 2B+\sin 2C}$

Similar results can be derived in complex numbers and co-ordinate geometry.

(iv) Further, if $\overrightarrow{OA}=\vec{a}; \overrightarrow{OB}=\vec{b}$,

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$$

Specifically if $\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ so that is (x_1, y_1, z_1) and

$\vec{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$ so that B is (x_2, y_2, z_2) then

$$-\vec{a}+\vec{b}=\overrightarrow{AB}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$$

and hence $|\vec{b}-\vec{a}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

which gives relatively the distance between $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.

13 PRODUCTS OF TWO VECTORS

To meet the needs of physical situations we have two kinds of products of two vectors:

(i) Resulting in scalar value and used in situations of work done by a force in a displacement; projection of a given length on another direction. This product is usually termed dot product or scalar product.

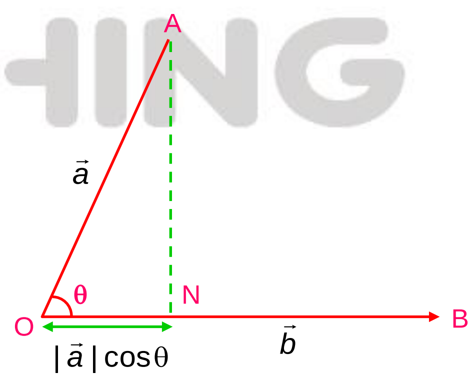
(ii) Resulting in a vector and used in situations of finding moment of a force about a point; or of finding the velocity of a particle having a rotating motion about a point with an angular velocity $\vec{\omega}$.

The two products with the related properties are stated:

13.1 THE SCALAR PRODUCT OR DOT PRODUCT

Let $\vec{a}$ and $\vec{b}$ be any two vectors, forming between the two, an angle θ $(0 \le \theta \le \pi)$. Then the scalar product or dot product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\cdot\vec{b}$ (Read $\vec{a}$ dot $\vec{b}$) and in value $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

Geometrically, it represents projection of a vector on the other.



Properties of the Scalar product

1. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ (Dot product is commutative)

2. $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ (Dot product is distributive)

3. $\vec{a}\cdot\vec{b}=0$ either when $|\vec{a}|=0$ or when $|\vec{b}|=0$ or when the vectors $\vec{a}$ and $\vec{b}$ are orthogonal. Thus for any two perpendicular vectors the dot product vanishes.

4. $\cos\theta=\left|\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right|$ where θ is the acute angle made by $\vec{a}$ with $\vec{b}$.

5. $\vec{a}\cdot\vec{a} = \vec{a}^{\,2} = |\vec{a}|^2$

6. For the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$

$$\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$$

$$\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{i}=\hat{i}\cdot\hat{k}=\hat{k}\,.\,\hat{i}=\hat{j}\cdot\hat{i}=\hat{i}\cdot\hat{j}=0$$

7. If $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ *and* $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, *then* $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$. so that

$$\cos\theta=\left|\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\,\sqrt{b_1^2+b_2^2+b_3^2}}\right|$$

8. The projection of $\vec{a}$ on another direction represented by $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

9. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, then

$$\vec{a}\cdot\hat{i}=a_1\,\hat{i}\cdot\hat{i}=a_1$$

Similarly $\vec{a}\cdot\hat{j}=a_2$ *and* $\vec{a}\cdot\hat{k}=a_3$ *so that*

$$\vec{a}=(\vec{a}\cdot\hat{i})\,\hat{i}+(\vec{a}\cdot\hat{j})\,\hat{j}+(\vec{a}\cdot\hat{k})\,\hat{k}$$

10. Work done by a force $\vec{F}$ in a displacement $\vec{AB}$ when the point of application of the force is displaced from *A* to *B*, is, $\vec{F}\cdot\vec{AB}=\vec{F}\cdot(\vec{OB}-\vec{OA})=\vec{F}$. (position vector of *B* – position vector of *A*)

13.2 THE VECTOR (OR CROSS) PRODUCT OF TWO VECTORS

Let $\vec{a}$ and $\vec{b}$ be any two vectors forming an angle θ ($0 \le \theta < \pi$). The vector product or cross product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ (Read as $\vec{a}$ cross $\vec{b}$); and this is a vector

(a) whose magnitude is $|\vec{a}||\vec{b}|\sin\theta$

(b) whose direction is perpendicular to both $\vec{a}$ and $\vec{b}$ such that looked from this direction the rotation from $\vec{a}$ to $\vec{b}$ through an angle $<\pi$ is anti-clockwise. It is written as $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\cdot\hat{n}$ where $\hat{n}$ is a unit vector in the direction of $\vec{a}\times\vec{b}$ i.e. in the direction perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

Properties of the vector product

1. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ (vector product is not commutative)
2. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ (vector product is distributive)
3. $\vec{a}\times\vec{b}=0$ either when $|\vec{a}|=0$ or when $|\vec{b}|=0$ or when the vectors have the same direction. Thus the vector product between two collinear vectors is zero.
4. A unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $(\vec{a}\times\vec{b})/|\vec{a}\times\vec{b}|$
5. $\vec{a}\times\vec{a}=0$ for any vector $\vec{a}$
6. For the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ taken along the coordinate axes

$$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0} \text{ while}$$
$$\hat{i}\times\hat{j}=-\hat{j}\times\hat{i}=\hat{k};\quad \hat{j}\times\hat{k}=-\hat{k}\times\hat{j}=\hat{i} \text{ and}$$
$$\hat{k}\times\hat{i}=-\hat{i}\times\hat{k}=\hat{j}$$

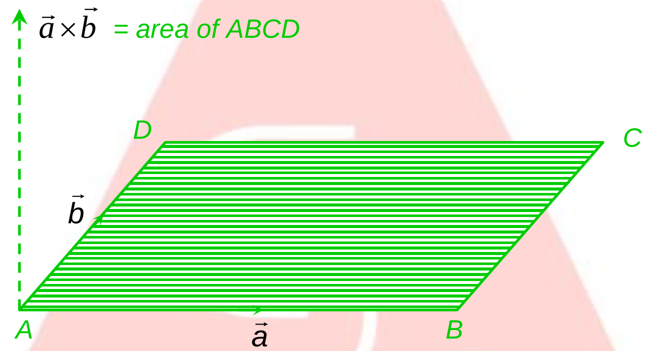
7. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $=\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

$$\vec{a}\times\vec{b}=\left(a_2b_3-a_3b_2\right)\hat{i}+\left(a_3b_1-a_1b_3\right)\hat{j}+\left(a_1b_2-a_2b_1\right)\hat{k}$$

or in an equivalent determinant form,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

8. $\vec{a}\times\vec{b}$ represents the vector area of the parallelogram whose adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



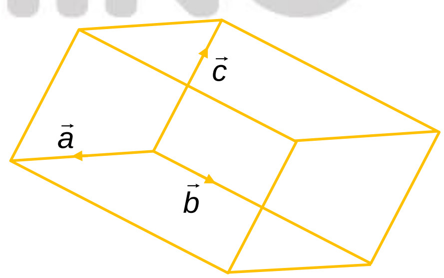
9. Let $\vec{F}$ be a force directed along a line. Let O be a point (origin). Let $\overrightarrow{OP}=\vec{r}$ be the position vector of any point P on the line of action of $\vec{F}$. Then $\vec{r}\times\vec{F}$ gives the moment of the force $\vec{F}$ about the point O.

10. Let $\vec{\omega}$ be the angular velocity of body rotating about an axis through O. If P be any point of the body with position vector $\overrightarrow{OP}=\vec{r}$, then $\vec{\omega}\times\vec{r}$ gives the velocity vector of P in the rotatory motion about the axis with an angular velocity $\vec{\omega}$.

14 PRODUCT OF THREE VECTORS

14.1 SCALAR TRIPLE PRODUCT

For any two vectors $\vec{b}$ and $\vec{c}$, $\vec{b}\times\vec{c}$ is a vector. This can be scalarly multiplied with a third vector $\vec{a}$ to give the scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$. This is a scalar whose value is the volume of a box having $\vec{a}$, $\vec{b}$, $\vec{c}$ as coterminous edges. Hence it is also written as $[\vec{a}\ \vec{b}\ \vec{c}]$ and in this sense is called the box product.



Properties of scalar triple product

1. $\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b})$

The value of the scalar triple product is unaltered for a cyclical change of the vectors in the product.

2. $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

The value of the scalar triple product is unaltered for a relative interchange of the positions of the dot and cross.

3. $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ either when $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar (or) when any one vector repeats in the triple product, like $\vec{a}\cdot(\vec{a}\times\vec{b})$, $\vec{a}\cdot(\vec{b}\times\vec{b})$ each one of these is zero.

4. For scalars, k, l and m

$$k\vec{a}\cdot(l\vec{b}\times m\vec{c}) = k\,l\,m\,\vec{a}\cdot(\vec{b}\times\vec{c})$$

5. In particular $\hat{i}.(\hat{j}\times\hat{k}) = 1$

6. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$,

$$\vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and hence the vanishing of this determinant is the condition for the coplanarity of the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ taken in terms of the $\hat{i}$, $\hat{j}$ and $\hat{k}$ components as shown above.

14.2 VECTOR TRIPLE PRODUCT

For three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ a product of the form $\vec{a}\times(\vec{b}\times\vec{c})$ or $(\vec{a}\times\vec{b})\times\vec{c}$ is called a vector triple product.

This is a vector, and the value depends upon the placement of the brackets. In fact $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector in the plane of $\vec{b}$ and $\vec{c}$ (the two placed in the brackets).

In value

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}\text{ ; and}$$

$$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{c}\cdot\vec{a})\,\vec{b} - (\vec{c}\cdot\vec{b})\,\vec{a}$$

15 SOME APPLICATIONS OF VECTORS

15.1 GEOMETRICAL APPLICATIONS

There is a wide application of vectors in geometry. Equation of line, plane, etc. are discussed in three dimensional geometry in details using vectors as well. Some results of geometry and trigonometry can be proved easily with the help of vectors, as shown in the following illustration.

15.2 BISECTOR OF AN ANGLE

If $\vec{a}$ and $\vec{b}$ are unit vectors along the sides of an angle, then $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are the vectors along internal and external bisectors of angle respectively.

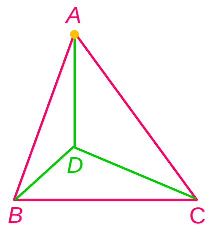
The bisector of angles between any two vectors $\vec{a}$ and $\vec{b}$ is given by the vector $\vec{r} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} \pm \frac{\vec{b}}{|\vec{b}|}\right)$

where $\lambda \in R$.

15.3 VOLUME OF A TETRAHEDRON

Volume of a tetrahedron $ABCD$

$$= \frac{1}{6}\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]$$



15.4 RECIPROCAL SYSTEM OF VECTORS

Let $\vec{a}, \vec{b}, \vec{c}$ be a system of non-coplanar vector, then the system of vectors $\vec{a}', \vec{b}', \vec{c}'$ which satisfy $\vec{a}.\vec{a}', \vec{b}.\vec{b}', \vec{c}.\vec{c}' = 1$ and $\vec{a}.\vec{b}', \vec{a}.\vec{c}', \vec{b}.\vec{a}' = \vec{c}.\vec{a}' = \vec{c}.\vec{b}' = 0$ is called a reciprocal system of vectors. Based on these condition $\vec{a}', \vec{b}', \vec{c}'$ are expressed as

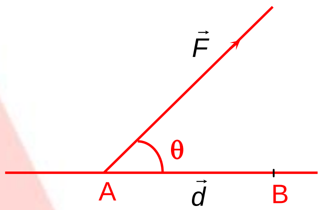
$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \quad \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \quad \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Also, $[\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}', \vec{b}', \vec{c}'] = 1$

16 APPLICATIONS OF VECTOR PRODUCT IN MECHANICS

16.1 Work done by a force

The work done by a force $\vec{F}$, acting on a body due to which displacement of body is $\vec{d}$, is given by



Work done (W) = (Magnitude of force in direction of displacement) × (distance moved)

$$= \left(|\vec{F}|\cos\theta\right)\left(|\vec{d}|\right) = \vec{F}.\vec{d}$$

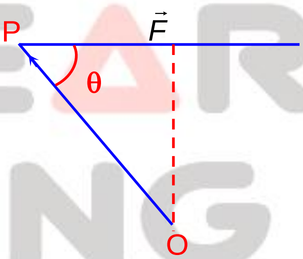
- The work done by a force is a scalar quantity.
- If a number of faces are acting on a body, then the sum of the works done by the separate forces is equal to the work done by the resultant force.

16.2 Moment of a force

(a) **About a point:**

Let a force $\vec{F}$ be applied at a point P. The moment of force $\vec{F}$ about a point O is defined as

$$\vec{M} = \overrightarrow{OP} \times \vec{F}$$



- Moment of force about a point is vector quantity.
- Moment is independent of selection of point P, infact P can be any point on the line of action of action of force $\vec{F}$.
- If several forces are acting through the point P, then the vector sum of the moments of the separate forces about O is equal to the moment of their resultant force about O.
- The moment of $\vec{F}$ about a point O measures the amount of $\vec{F}$ to turn the body about point O. If tendency of rotation is in anticlockwise direction, the moment is positive, otherwise it is negative.

(b) **About a line:**

Let $\vec{F}$ be any given force, acting at a point P and L be any directed line segment. The moment of force $\vec{F}$ about line L is defined as

$$M_a = \left(\overrightarrow{OP} \times \vec{F}\right).\hat{a}$$

where $\hat{a}$ is a unit vector in the direction of line and O is any point on the line.

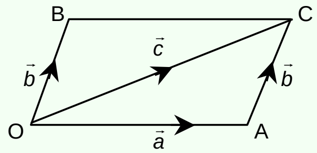
- Moment about a line is a scalar quantity.
- Moment of $\vec{F}$ about the line L is the projection along L, of the vector moment of the force $\vec{F}$ about any point on the L.

MIND MAP

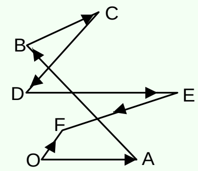
VECTOR ALGEBRA

LAW OF ADDITION OF VECTORS

- The parallelogram law of addition is $\vec{a}+\vec{b}=\vec{c}$ $(\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OC})$



- The triangle law is $\vec{a}+\vec{b}=\vec{c}$ $(\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC})$
- The polygon law of addition is $\overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EF}=\overrightarrow{OF}$



ORTHOGONAL SYSTEM OF VECTORS

Any point $P(x, y, z)$ has position vector $\overrightarrow{OP}=x\hat{i}+y\hat{j}+z\hat{k}$ where $\hat{i},\hat{j},\hat{k}$ are three mutually perpendicular unit vectors creating an orthogonal system.

$$|\overrightarrow{OP}|=\sqrt{x^2+y^2+z^2}$$

Unit vector in direction of $\overrightarrow{OP}$ is $\dfrac{\overrightarrow{OP}}{|\overrightarrow{OP}|}$

$$=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$= l\hat{i}+m\hat{j}+n\hat{k}$ where l, m and n are known as direction cosines.

PRODUCT OF TWO VECTORS

For any two vectors

$$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$$

and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

- Scalar product :

$$\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$$

also $\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$. It represents projection of a vector on the other.

- $\vec{a}.\vec{b}=0 \Rightarrow$ either $|\vec{a}|=0$ and $|\vec{b}|=0$ or they are orthogonal.
- Vector product represents area of parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\hat{n}$ where $\hat{n}$ is unit vector perpendicular to plane of $\vec{a}$ and $\vec{b}$.
- $\vec{a}\times\vec{b}=0 \Rightarrow$ either $|\vec{a}|=0$ or $|\vec{b}|=0$ or $\vec{a}$ is collinear with $\vec{b}$

LINEAR COMBINATION OF VECTORS

- Fundamental theorem in two dimension: Any vector $\vec{r}$, coplanar with non-collinear vectors $\vec{a}$ and $\vec{b}$ can be uniquely expressed as $\vec{r}=x\vec{a}+y\vec{b}$.
- Fundamental theorem in three dimension: Any vector $\vec{r}$ can be uniquely expressed as a linear combination of three non-coplanar vectors $\vec{a},\vec{b}$ and $\vec{c}$ as

  $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}$.
- Linearly dependent and independent vectors. A system of vectors $\vec{a}_1,\vec{a}_2,....\vec{a}_n$ is said to be linearly dependent if there exist a system of scalars $\lambda_1,\lambda_2....\lambda_n$ (not all zero) such that $\lambda_1\vec{a}_1+\lambda_2\vec{a}_2+....+\lambda_n\vec{a}_n=0$. They are linearly depend if every such relation implies $\lambda_1=\lambda_2=\lambda_n=0$.
- Collinearity: Three points $\vec{a},\vec{b},\vec{c}$ are collinear $\Leftrightarrow$ $x\vec{a}+y\vec{b}+z\vec{c}=0$ where $x+y+z=0$.
- Coplanarity: Four points $\vec{a},\vec{b},\vec{c},\vec{d}$ are coplanar $\Leftrightarrow$ $x\vec{a}+y\vec{b}+z\vec{c}+w\vec{d}=0$ where $x+y+w+z=0$

PRODUCT OF THREE VECTORS

For any three vectors $\vec{a},\vec{b},\vec{c}$

- Scalar triple product: represents the volume of parallelepiped with adjacent sides, and is given by

$$[\vec{a}\ \vec{b}\ \vec{c}]=\vec{a}.(\vec{b}\times\vec{c})$$

- Vector triple product: represents a vector $\vec{r}=\vec{a}\times(\vec{b}\times\vec{c})$, in the plane of $\vec{b}$ and $\vec{c}$ as

$$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$$

SOME IMPORTANT RESULTS

- Angle bisectors of $\vec{a}$ and $\vec{b}$ is given by vector $\lambda\left(\dfrac{\vec{a}}{|\vec{a}|}\pm\dfrac{\vec{b}}{|\vec{b}|}\right)$.
- Volume of tetrahedron

$$=\frac{1}{6}|[\vec{a}\,\vec{b}\,\vec{c}]|.$$

- Reciprocal system of vectors

$$\vec{a}'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]},\ \vec{b}'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]},\ \vec{c}'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$$